Solution to Exercise 2: Basic reproduction number R_0 estimation for models SIP and SIPD

Method of calculation

As in Lesson 5, NGM method was applied to construct expressions of *Ro* for SIP and SIPD models. The necessary steps are the following:

1. Find the steady state for the population with no infected individuals or no infectious elements.
2. Linearize the equations around a population with some number of susceptibles ($S = N$), where N is the initial number of susceptibles in the population, and small numbers of infected ($I \ll N$) or infectious elements ($D, P \ll N$).
3. Consider the subset of the equations that involve the infectious processes. That is, ignore the $\frac{dS}{dt}$ equation.
4. Separate the terms in the remaining equations into two parts: transmission, represented by a matrix $\mathbf{T}$, and transistions, represented by a matrix Σ.
5. Construct the next-generation matrix (NGM) for the large domain ($\mathbf{K_L} = -\mathbf{T} \cdot \Sigma^{-1}$).
6. Calculate the eigenvalues for $\mathbf{K_L}$. The dominant eigenvalue is *Ro*.

R_0 for the SIP model

As described in the solution of Exercise 1, this model represents disease transmission by contact of susceptibles with particles released by infected individuals. The particles disappear at a rate r which can be deactivation, dilution, decay or other losses.

$$\frac{d\,S}{dt} = -\beta\, P\, S$$
$$\frac{d\,I}{dt} = \beta\, P\, S - m\, I$$
$$\frac{d\,P}{dt} = c\, I - r\, P$$

The equation for a non-infected population ($I = P = 0$) is $\frac{d\,S}{dt} = 0$. The population is constant at the initial population level of N.

The linearized equations around the steady state have $S = N$ and $I, D \ll N$. The resulting linearized equations for the infection processes are

$$\frac{d\,I}{dt} = \beta\, P\, N - m\, I$$
$$\frac{d\,P}{dt} = c\, I - r\, P$$

The transmission vector is $\vec{x} = (P, I)$. Transmission and transition matrices along with the inverse of the transition matrix are

$$\mathbf{T} = \begin{pmatrix} 0 & c \\ \beta\, N & 0 \end{pmatrix} \quad \Sigma = \begin{pmatrix} -r & 0 \\ 0 & -m \end{pmatrix} \text{ and } \Sigma^{-1} = \begin{pmatrix} -1/r & 0 \\ 0 & -1/m \end{pmatrix}$$

The large domain NGM is

$$\mathbf{K_L} = -\mathbf{T} \cdot \Sigma^{-1} = \begin{pmatrix} 0 & c/m \\ \beta\, N/r & 0 \end{pmatrix}$$

The eigenvalues are the solution of the determinant equation similar to above (basically replacing the diagonal zeros with $= \lambda$ and setting the determinant to zero). The resulting equation is

$$\lambda^2 - \beta\, N\, c/m\, r = 0 \text{ with solution } \lambda = \sqrt{\frac{c}{r}\frac{\beta\, N}{m}} = Ro.$$

This formulation shows that Ro is the geometric mean of ratios of the production and destruction of the two step process creating the infection.

R_0 for the SIPD model

As described in the solution of Exercise 1, this model represents disease transmission by contact of susceptibles with particles released by dead individuals. Both dead individuals and infectious particles are removed by rates d and r, respectively.

$$\frac{d\,S}{dt} = -\beta\,P\,S$$
$$\frac{d\,I}{dt} = \beta\,P\,S - m\,I$$
$$\frac{d\,D}{dt} = m\,I - d\,D$$
$$\frac{d\,P}{dt} = c\,D - r\,P$$

The equation for a non-infected population ($I = D = P = 0$) is $\frac{d\,S}{dt} = 0$. The population is constant at the initial population level of N.

The linearized equations around the steady state have $S = N$ and $I, D \ll N$. The resulting linearized equations for the infection processes are

$$\frac{d\,P}{dt} = c\,D - r\,P$$
$$\frac{d\,D}{dt} = m\,I - d\,D$$
$$\frac{d\,I}{dt} = \beta\,P\,N - m\,I$$

The transmission vector is $\vec{x} = (P, D, I)$. Transmission and transition matrices along with the inverse of the transition matrix are

$$\mathbf{T} = \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & m \\ \beta\,N & 0 & 0 \end{pmatrix} \quad \Sigma = \begin{pmatrix} -r & 0 & 0 \\ 0 & -d & 0 \\ 0 & 0 & -m \end{pmatrix} \text{ and } \Sigma^{-1} = \begin{pmatrix} -1/r & 0 & 0 \\ 0 & -1/d & 0 \\ 0 & 0 & -1/m \end{pmatrix}$$

The large domain NGM is

$$\mathbf{K_L} = -\mathbf{T} \cdot \Sigma^{-1} = \begin{pmatrix} 0 & c/d & 0 \\ 0 & 0 & 1 \\ \beta\,N/r & 0 & 0 \end{pmatrix}$$

The eigenvalues are the solution of the determinant equation similar to above (basically replacing the diagonal zeros with $= \lambda$ and setting the determinant to zero). The determinant is calculated by reducing by minors as

$$\begin{vmatrix} -\lambda & c/d & 0 \\ 0 & -\lambda & 1 \\ \beta\,N/r & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} - \frac{c}{d} \begin{vmatrix} 0 & 1 \\ \beta\,N/r & -\lambda \end{vmatrix} = -\lambda^3 + \frac{c}{r}\frac{\beta\,N}{d} = 0$$

The solution is

$$\lambda = \sqrt[3]{\frac{c}{d}\frac{\beta\,N}{r}} = Ro.$$

Changing the expression slightly shows again that Ro is the geometric mean of the creation and removal rates:

$$Ro = \sqrt[3]{\frac{c}{r}\frac{\beta\,N}{d}}$$