K-V BLOKEA: Oinarrizko integralen kalkulua- Integral trigonometrikoak

Integral trigonometrikoak ebazterako orduan gehien erabiltzen diren berdintza trigonometrikoak honakoak dira:

Formula Trigonometrikoak:

$$\sin^2 a + \cos^2 a = 1 \quad (1)$$

$$\cos^2 a = \frac{1 + \cos 2a}{2} \quad (2)$$

$$\sin^2 a = \frac{1 - \cos 2a}{2} \quad (3)$$

$$\cos(2a) = \cos^2 a - \sin^2 a \quad (4)$$

$$\sin(2a) = 2 \cdot \sin a \cdot \cos a \quad (5)$$

$$\cos(a \pm b) = \cos a \cdot \cos b \mp \sin a \cdot \sin b \quad (6)$$

$$\sin(a \pm b) = \sin a \cdot \cos b \pm \sin b \cdot \cos a \quad (7)$$

$$\cos a \cdot \cos b = \frac{\cos(a+b) + \cos(a-b)}{2} \quad (8)$$

$$\sin a \cdot \sin b = \frac{\cos(a-b) - \cos(a+b)}{2} \quad (9)$$

$$\sin a \cdot \cos b = \frac{\sin(a+b) + \sin(a-b)}{2} \quad (10)$$

1. ARIKETA:

Kalkula ezazu hurrengo integrala:

$$\int \sin 6x \cdot \sin 2x \, dx$$

Ebazpena:

Ondorengo berdintza erabiliz:

$$\sin a \cdot \sin b = \frac{\cos(a-b) - \cos(a+b)}{2}$$

$$\int \sin 6x \cdot \sin 2x \, dx = \int \frac{\cos 4x - \cos 8x}{2} dx = \frac{1}{2}\int \cos 4x \, dx - \frac{1}{2}\int \cos 8x \, dx =$$

$$\frac{1}{2}\frac{\sin 4x}{4} - \frac{1}{2}\frac{\sin 8x}{8} + C = \frac{1}{8}\sin 4x + \frac{1}{16}\sin 8x + C$$

2. ARIKETA:

Kalkula ezazu hurrengo integrala:

$$\int \cos^6 x\, dx$$

Ebazpena:

Honako berdintza hau erabiliz:

$$\cos^2 a = \frac{1+\cos 2a}{2}$$

$$\begin{aligned}\int \cos^6 x\, dx &= \int (\cos^2 x)^3\, dx = \int \left(\frac{1+\cos 2x}{2}\right)^3 dx \\ &= \int \frac{1+3\cos 2x + 3\cos^2 2x + \cos^3 2x}{8} dx = \\ &= \frac{1}{8}\int dx + \frac{3}{8}\int \cos 2x\, dx + \frac{3}{8}\int \cos^2 2x\, dx + \frac{1}{8}\int \cos^3 2x\, dx = \\ &= \frac{x}{8} + \frac{3}{8}\frac{\sin 2x}{2} + C_3 + \frac{3}{8}\underbrace{\int \cos^2 2x\, dx}_{(A)} + \frac{1}{8}\underbrace{\int \cos^3 2x\, dx}_{(B)}\end{aligned}$$

(A) $\int \cos^2 2x\, dx$ kalkulatzeko berriro ere $\cos^2 a = \frac{1+\cos 2a}{2}$ formula aplikatuz:

$$\int \cos^2 2x\, dx = \int \frac{1+\cos 4x}{2} dx = \frac{x}{2} + \frac{\sin 4x}{8} + C_1$$

(B) $\int \cos^3 2x\, dx$ integrala kalkulatzeko $\sin^2 a + \cos^2 a = 1$ berdintza erabiliz:

$$\begin{aligned}\int \cos^3 2x\, dx &= \int \cos 2x \cos^2 2x\, dx = \int \cos 2x\,(1-\sin^2 2x) dx \\ &= \int \cos 2x\, dx - \int \cos 2x \sin^2 2x\, dx = \frac{\sin 2x}{2} - \frac{1}{6}\sin^3 2x + C_2\end{aligned}$$

Ondorioz, integral hauek hasierakoan ordezkatuz:

$$\int \cos^6 x\, dx = \frac{x}{8} + \frac{3}{8}\frac{\sin 2x}{2} + \frac{3}{8}\left(\frac{x}{2} + \frac{\sin 4x}{8}\right) + \frac{1}{8}\left(\frac{\sin 2x}{2} - \frac{1}{6}\sin^3 2x\right) + C =$$

$$\frac{x}{8} + \frac{3}{16}\sin 2x + \frac{3}{16}x + \frac{3}{16}\sin 4x + \frac{1}{16}\sin 2x - \frac{1}{48}\sin^3 2x + C =$$

$$\frac{5x}{16} + \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{48}\sin^3 2x + C$$

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Aldagai aldaketa orokorra:

Integral trigonometrikoak ebazteko era oso zabaldu bat aldagai aldaketa orokorra da.

Aldagai aldaketa orokorra honako hau da:

$$\tan\left(\frac{x}{2}\right) = t \Longrightarrow x = 2\arctan(t) \Longrightarrow dx = \frac{2dt}{1+t^2}$$

Berdintza honetatik ondorengoa berdintzak lor daitezke:

$$\sin(x) = \frac{2t}{t^2+1};\ \cos(x) == \frac{1-t^2}{t^2+1};\ \tan x = \frac{2t}{1-t^2}$$

Aldagai aldaketa honi esker integral trigonometrikoak integral arrazionalak bilakatzen dira.

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3. ARIKETA:

Kalkula ezazu hurrengo integrala:

$$\int \frac{dx}{4-2\cos(x)}$$

Ebazpena:

Aldagai aldaketa orokorra erabiliz:

$$\cos(x)=\frac{1-t^2}{t^2+1},\quad dx=\frac{2dt}{1+t^2}$$

Integralari aplikatuz:

$$\int \frac{dx}{4-2\cos(x)}=\int \frac{2dt}{1+t^2}\frac{1}{4-2\cdot\frac{1-t^2}{t^2+1}}=\int \frac{2dt}{1+t^2}\frac{1}{\frac{4(t^2+1)-2(1-t^2)}{t^2+1}}=$$

$$\int \frac{2dt}{4t^2+4-2+2t^2}=\int \frac{2dt}{6t^2+2}=\int \frac{dt}{3t^2+1}=\int \frac{dt}{\left(\sqrt{3}t\right)^2+1}=\frac{1}{\sqrt{3}}\int \frac{\sqrt{3}}{\left(\sqrt{3}t\right)^2+1}=$$

$$\frac{1}{\sqrt{3}}\arctan\left(\sqrt{3}t\right)+C$$

Bukatzeko, aldagai aldaketa desegiten:

$$\int \frac{dx}{4-2\cos(x)}=\frac{1}{\sqrt{3}}\arctan\left(\sqrt{3}\tan\left(\frac{x}{2}\right)\right)+C$$

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4. ARIKETA:

Kalkula ezazu hurrengo integrala:

$$\int \frac{\tan x}{1-\cos(x)} \cdot dx$$

Ebazpena:

Aldagai aldaketa orokorra erabiliz:

$$\tan x = \frac{2t}{1-t^2},\ \cos(x) = \frac{1-t^2}{t^2+1}, dx = \frac{2dt}{1+t^2}$$

Integralari aplikatuz:

$$\int \frac{\tan x}{1-\cos(x)} \cdot dx = \int \frac{2t}{1-t^2} \frac{1}{1-\frac{1-t^2}{t^2+1}} \cdot \frac{2dt}{1+t^2} = \int \frac{4t}{(1-t^2)\,\frac{[t^2+1-(1-t^2)](1+t^2)}{t^2+1}} dt =$$

$$\int \frac{4t}{(1-t^2)(t^2+1-1+t^2)} dt = \int \frac{4t}{(1-t^2)(2t^2)} dt = 2\int \frac{1}{(1-t^2)t} dt =$$

Integral arrazionala lortu dugu, integral hau ebatziz:

$$\frac{1}{(1-t^2)t} = \frac{1}{t(1-t)(1+t)} = \frac{A}{t} + \frac{B}{1-t} + \frac{C}{1+t} \Rightarrow$$

$$\frac{1}{(1-t^2)t} = \frac{A(1-t)(1+t) + Bt(1+t) + Ct(1-t)}{t(1-t)(1+t)} \Rightarrow$$

$$1 = A(1-t)(1+t) + Bt(1+t) + Ct(1-t)$$

$$t = 0 \Rightarrow 1 = A$$

$$t = 1 \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$$

$$t = -1 \Rightarrow 1 = -2C \Rightarrow C = -\frac{1}{2}$$

Ondorioz,

$$2\int \frac{1}{(1-t^2)t} dt = 2\int \frac{dt}{t} + 2\int \frac{1/2}{1-t} dt + 2\int \frac{-1/2}{1+t} dt =$$

$$2\int \frac{dt}{t} + \int \frac{dt}{1-t} - \int \frac{dt}{1+t} = 2\ln|t| - ln|1-t| - ln|1+t| + C$$

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Bukatzeko, aldagai aldaketa desegiten:

$$\int \frac{\tan x}{1-\cos(x)}\cdot dx = 2\ln\left|\tan\left(\frac{x}{2}\right)\right| - \ln\left|1-\tan\left(\frac{x}{2}\right)\right| - \ln\left|1+\tan\left(\frac{x}{2}\right)\right| + C$$

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