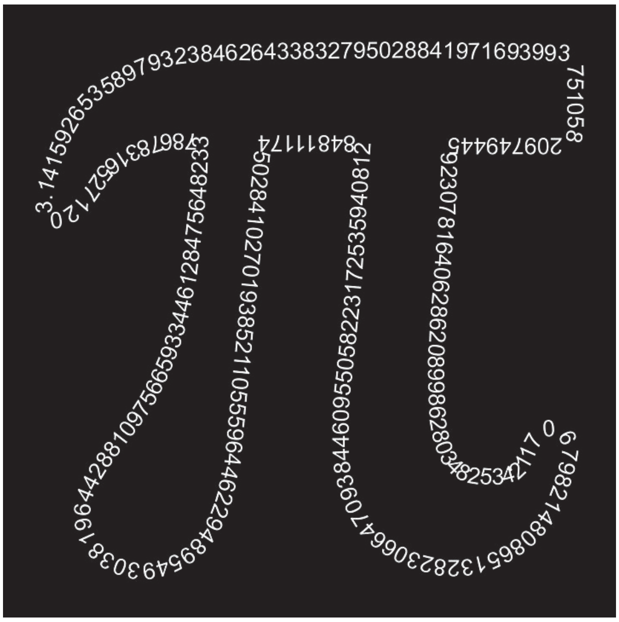
MATHS BASIC COURSE FOR UNDERGRADUATES



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TEST 1. STATEMENTS

Exercise 1. Prove by induction that the statement $2^n > 2n+1$ fulfills, for any $n \geq 3$.

Exercise 2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by parts as follows:

$$f(x) = \begin{cases} 2, & \text{when } x < 0 \\ 1, & \text{when } 0 \leq x \leq 3 \\ x-4, & \text{when } x > 3 \end{cases}$$

(i) Is the function f injective? Is it surjective?
(ii) Calculate $f((1,3))$ and $f^{-1}((0,1))$.

Exercise 3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be two functions defined as follows:

$$f(x) = \begin{cases} 1, & \text{when } x \leq -1 \\ x+2, & \text{when } x > -1 \end{cases}$$

and

$$g(x) = 2x, \text{ for any } x \in \mathbb{R}.$$

(i) Analyze whether the composition function $f \circ g$ is injective and/or surjective.
(ii) In case it is possible, calculate $(f \circ g)^{-1}(1)$ and $(f \circ g)^{-1}(2)$.

Exercise 4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by parts as follows:

$$f(x) = \begin{cases} -x, & \text{if } x < -2 \\ x^2-1, & \text{if } -2 \leq x \leq 2 \\ x, & \text{if } x > 2 \end{cases}$$

(i) Analyze whether the function f is injective and/or surjective.
(ii) Calculate $f((0,3])$ and the inverse image of the set $[-3,3]$ by f: $f^{-1}([-3,3])$.
(iii) Consider the function g over $\mathbb{R}$ defined by the expression $g(x) = |x|$. Calculate the composition function $g \circ f$, and the inverse image of the set $\{1\}$ by the function $g \circ f$: $(g \circ f)^{-1}(1)$.

Exercise 5. Assume that given the functions $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, the function $g \circ f$ is surjective. Then, prove that g is surjective, but that the converse implication does not fulfill.

Exercise 6. Prove that the square of any integer number is of type: $5k$, $5k+1$ or $5k+4$, for some k integer number.

Exercise 7. i) Assume that $a = bq + r$, being $a, q \in \mathbb{Z}$, $b, r \in \mathbb{N}$ and $0 \leq r < b$. Prove that $\gcd(a,b) = \gcd(b,r)$.

iii) Using the previous item (i), calculate $\gcd(102,44)$, and solve the equation $44x + 102y = 2$, for some $(x,y) \in \mathbb{Z} \times \mathbb{Z}$.

Exercise 8. Assume that a, b are two integer numbers which are coprime, and that c is also an integer number. Prove that $\gcd(ac,b) = \gcd(c,b)$. Latter on, using this result calculate $\gcd(5000, 31768)$.