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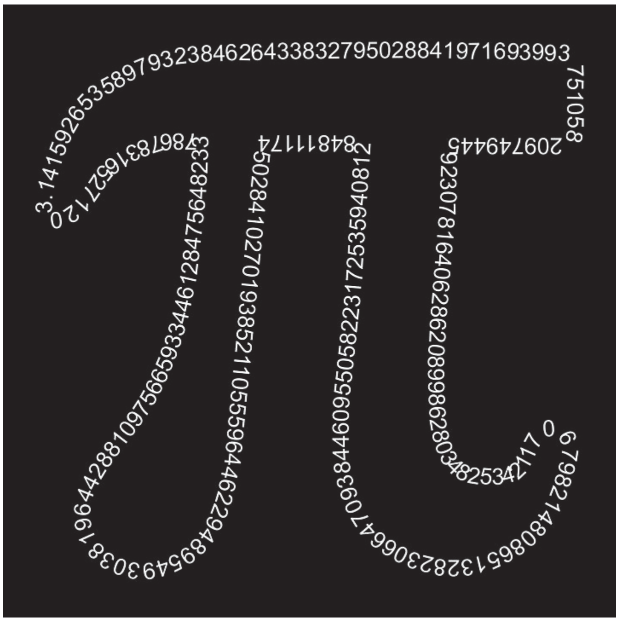
MATHS BASIC COURSE FOR UNDERGRADUATES



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SOLUTIONS: 5th SUBJECT. CONGRUENCES

SOLUTION EXERCISE 1: In congruence language, we have to find an integer number r comprehended between 0 and 12, for which $n \equiv r(\text{mod } 12)$, in other words, we have to reduce n to modulo 12. First of all, $4! = 24 \equiv 0(\text{mod } 12)$, and consequently if $k \geq 4$,

$$k! = k(k-1)\dots 6 \cdot 5 \cdot 4! \equiv k(k-1)\dots 6 \cdot 5 \cdot 0 \equiv 0(\text{mod } 12).$$

Thus, $n \equiv 1! + 2! + 3!(\text{mod } 12)$, and therefore $n \equiv 9(\text{mod } 12)$.

SOLUTION EXERCISE 2: We have to prove that the remainder of the division of $5^{2k}+3\cdot 2^{5k-2}$ by 7 is 0. By applying congruences' properties and since $3 \equiv -2^2(\text{mod } 7)$ is fulfilled, we get the following congruences modulo 7:

$$5^{2k} + 3 \cdot 2^{5k-2} \equiv 5^{2k} + (-2^2) \cdot 2^{5k-2} \equiv 5^{2k} - 2^{5k} \equiv 25^k - 32^k(\text{mod } 7).$$

On the other hand, since $25 \equiv 4(\text{mod } 7)$ and $32 \equiv 4(\text{mod } 7)$, by applying congruences' properties, we obtain that

$$5^{2k} + 3 \cdot 2^{5k-2} \equiv 25^k - 32^k \equiv 4^k - 4^k \equiv 0(\text{mod } 7).$$

SOLUTION EXERCISE 3: Write n in decimal form,

$$n = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \cdots + a_k \cdot 10^k,$$

where a_i satisfies $0 \leq a_i \leq 9$. Since $10 \equiv 1(\text{mod } 9)$, applying congruences's properties, $10^i \equiv 1(\text{mod } 9)$ and $a_i \cdot 10^i \equiv a_i(\text{mod } 9)$. Thus, $n \equiv a_1 + a_2 + \cdots + a_n(\text{mod } 9)$.

SOLUTION EXERCISE 4: Since $614 \equiv 2(\text{mod } 17)$ $(614 = 36 \cdot 17 + 2)$, then $614^{6943} \equiv 2^{6943}(\text{mod } 17)$. Since $17 \nmid 2$, using Fermat's Little Theorem, we have that $2^{16} \equiv 1(\text{mod } 17)$. Now, since $6943 = 433 \cdot 16 + 15$,

$$2^{6943} \equiv 2^{433\cdot 16+15} \equiv (2^{16})^{433} 2^{15} \equiv 1^{433} 2^{15} \equiv 2^{15}(\text{mod } 17).$$

Finally, since $2^4 \equiv 16 \equiv -1(\text{mod } 17)$, then

$$2^{15} \equiv 2^{4\cdot 3+3} \equiv (2^4)^3 2^3 \equiv (-1)^3 2^3 \equiv -8 \equiv 9(\text{mod } 17).$$

Thus, the remainder that we were looking for is 9.

SOLUTION EXERCISE 5: If $x \in \mathbb{Z}$ is a solution for this linear congruence, then $13x = 2 + 31q$, for some $q \in \mathbb{Z}$. Observe that $\gcd(13, 31) = 1$, and then by applying Bezout's identity, there exist two integer numbers s, t for which $1 = 13s + 31t$. Thus,

$13s \equiv 1(\bmod 31)$, and multiplying the previous congruence by 2, we obtain $13(2s) \equiv 2(\bmod 31)$. This is, $x = 2s$ ($s \in \mathbb{Z}$) is a solution for the initial congruence.

SOLUTION EXERCISE 6: By Fermat's Little Theorem, the congruence $x^{p-1} \equiv 1(\bmod p)$, or equivalently the congruence $x^{p-1} - 1 \equiv 0(\bmod p)$ has $p-1$ different solutions. To be more precise, the solutions of the previous congruence are: $1, 2, \ldots, p-1$. Thus,

$$x^{p-1} - 1 \equiv (x-1)(x-2)\ldots(x-(p-1))(\bmod p).$$

Now, since any of those values of x fulfills this congruence, by taking $x = 0$ we obtain that:

$$-1 \equiv (-1)(-2)\ldots(-(p-1))(\bmod p) \equiv (-1)^{p-1} 1 \cdot 2 \ldots (p-1)(\bmod p),$$

this is, $(-1)^{p-1} \cdot (p-1)! + 1 \equiv 0(\bmod p)$. In particular, if $p = 2$, we get $-1 + 1 \equiv 0(\bmod 2)$, which is obvious, and if p is an odd number, we get $(p-1)! + 1 \equiv 0(\bmod p)$.