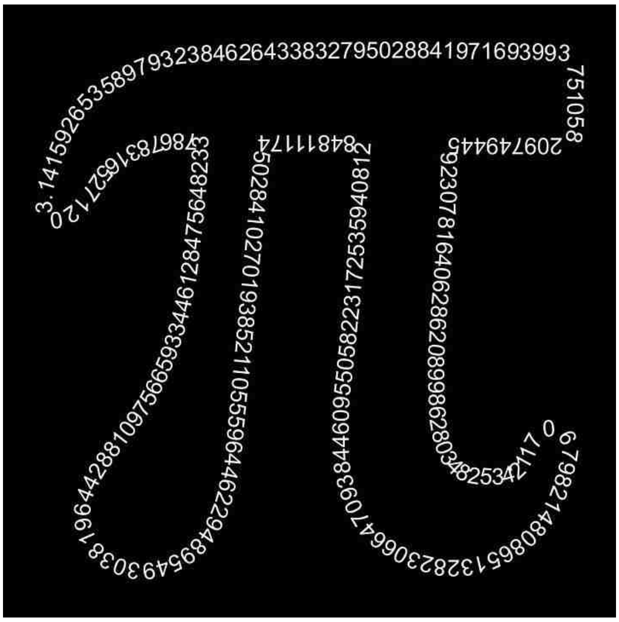
MATHS BASIC COURSE FOR UNDERGRADUATES



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8th SUBJECT: ELEMENTARY LINEAR ALGEBRA

1 Elementary Linear Algebra

1.1 Preliminary concepts:

(i) A set G with an inner operation $*$, is called a group, if the operation $*$ satisfies the associative property, there exists a neutral element and if for any element of G there exists an inverse respect to that operation. In addition to this, if $*$ satisfies the commutative property, then the group G is said to be a commutative group.

(ii) A set A with two inner operations (addition and multiplication) is said to be a ring if the following conditions hold: (a) $(A, +)$ is a commutative group; (b) $(.)$ is associative, and (c) the multiplication is distributive (in both sides) respect to the addition; i.e for any $x, y, z \in A$ the equality $x.(y + z) = x.y + x.z; (y + z).x = y.x + z.x$ holds. In addition to this, if $(.)$ is commutative, the ring $(A, +, .)$ is said to be a commutative ring, and if there exists a neutral element (1) respect to $(.)$, then the ring $(A, +, .)$ is said to be a ring with identity.

(iii) A field K is a commutative ring for which the set formed by all the non-null elements has the structure of a group respect to the multiplication.

(iv) Let K be a field and V be a set. If $+$ is an inner operation in V, and $\cdot : K \times V \longrightarrow V$ is an scalar multiplication between elements $\lambda \in K$ and elements $v \in V$ satisfying the following properties:

  (a) $(V, +)$ has a commutative group structure,

  (b) i. $\lambda \cdot (v_1 + v_2) = \lambda \cdot v_1 + \lambda \cdot v_2, \forall \lambda \in K, \forall v_1, v_2 \in V$

   ii. $(\lambda_1 + \lambda_2) \cdot v = \lambda_1 \cdot v + \lambda_2 \cdot v, \forall \lambda_1, \lambda_2 \in K, \forall v \in V$

   iii. $\lambda_1 \cdot (\lambda_2 \cdot v) = (\lambda_1 \cdot \lambda_2) \cdot v, \forall \lambda_1, \lambda_2 \in K, \forall v \in V$

   iv. $1 \cdot v = v, \forall v \in V$,

then V is said to be a vector space over the field K or it is said that V is a K-vector space or simply V is a vector space, and the elements of V are called vectors and the elements of K scalars.

(v) If V is a K-vector space and W a subset of V, which with the operations inherited from V has a K-vector space structure, then W is said to be a vector K-subspace of V. It is denoted by $W \leq V$.

Definition. *Let V be a K-vector space. A linear combination of the vectors $v_1, \dots, v_n \in V$ is a vector v of V that can be written as follows:*

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n,$$

being $\lambda_1, \lambda_2, \dots, \lambda_n \in K$.

Definition. *Let V be a K-vector space and S be a subset of V. Then,*

(i) *S is said to be linearly dependent if there exists any linear combination of some vectors of S equals to the null vector 0 that does not force all the scalar λ_i to be 0 in K. Symbolically, $\exists \lambda_1, \dots, \lambda_n \in K$ not all equal to 0, and $\exists v_1, \dots, v_n \in S$ such that $\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$.*

(ii) *Otherwise, S is said to be an independent system in V. In other words, in order to prove that S is an independent system the following statement has to be proved:*

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = 0, v_i \in S \text{ and } \lambda_i \in K \Longrightarrow \lambda_1 = \cdots = \lambda_n = 0.$$

Definition. *Let V be a vector space and S be a subset of V. Then,*

(i) *S is said to be a generating system of V, if the set formed by all the possible combinations of elements of S coincides with the whole set V. In that case, we say that the set S generates V.*

(ii) *V is said to be finitely generated if V can be generated by a finite set of vectors of V.*

Definition. *Let V be a vector space and β a subset of V. The set β is said to be a basis of V if it is linearly independent and it generates V.*

Theorem. *In a finitely generated vector space V, there always exist bases and all the bases have the same cardinal.*

Definition. *Let V be a K-vector space. The dimension of V is defined as the common cardinal of all the bases of V, and it is denoted by dim $_K V$ or by dim V.*

1.2 Matrices. Addition and scalar multiplication of matrices. Matrix multiplication. Invertible matrices

Definition. *Let K be a field and $n, m \in \mathbb{N}$. A matrix of size $n \times m$ over the field K is a table of n rows and m columns whose elements belong to the field K. The term of the* *matrix A, which is in the i-th row and in the j-th column, simultaneously, is denoted by a_{ij}. It means,*

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} = (a_{ij})_{i\in\{1,\dots,n\},j\in\{1,\dots,m\}}.$$

The set of all the matrices of this type is denoted by $M_{n,m}(K)$. Any two matrices A and B of size $n \times m$ are equal if and only if $\forall i \in \{1,\dots,n\}$ and $\forall j \in \{1,\dots,m\}$, $a_{ij} = b_{ij}$. A matrix A is said to be a square matrix when $n = m$, and the set formed by all the matrices of this type is denoted by $M_n(K)$.

Definition. *Let $A, B \in M_{n,m}(K)$ and $\lambda \in K$. We define the following operations between matrices:*

(i)

$$\lambda A = \lambda \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1m} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda a_{n1} & \lambda a_{n2} & \cdots & \lambda a_{nm} \end{pmatrix}$$

(ii)

$$\begin{aligned} A + B &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2m}+b_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nm}+b_{nm} \end{pmatrix} \end{aligned}$$

(iii) *If $A \in M_{n,m}(K), B \in M_{m,t}(K)$ (being $n, m, k \in \mathbb{N}$),*
$A \cdot B = C = (c_{ij})_{i\in\{1,\dots,n\},j\in\{1,\dots,t\}}$ *such that*

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}, \forall i \in \{1,\dots,n\}, \forall j \in \{1,\dots,t\}.$$

Definition. *Let $A \in M_{n,m}(K)$. The transposed matrix of A, denoted by A^t, corresponds to the following matrix,*

$$A^t = (b_{i,j})_{i\in\{1,\ldots,m\},j\in\{1,\ldots,n\}} \text{ such that } b_{ij} = a_{ji}, \forall i \in \{1,\ldots,m\}, \forall j \in \{1,\ldots,n\}.$$

Definition. *Let $A \in M_n(K)$ be a square matrix. By definition, the matrix A is said to be invertible if there exists a matrix $B \in M_n(K)$ such that $AB = BA = I_n$ (identity matrix of order n). In that case, B is denoted by A^{-1}, and it is called the inverse matrix of A. In the case the matrix A is invertible, its inverse matrix is unique, and it is said that the matrix A belongs to the set $\mathrm{GL}_n(K)$.*

In the following, we will define what the rank of matrix is (defining first the rank of a matrix by rows or the rank of a matrix by columns). To do this, let $A \in M_{n,m}(K)$ be a matrix and $A_1, A_2, \ldots, A_n$ the rows of the matrix A, and $A^1, A^2, \ldots, A^m$ the columns of the matrix A, respectively. In particular, the rows $A_1, A_2, \ldots, A_n$ are m-tuples, and they can be viewed as vectors of m components, and analogously the thansposes of the columns $A^1, A^2, \ldots, A^m$ are n-tuples, and they can be viewed as vectors of n components.

Definition. (i) *The rank of the matrix A by rows is the dimension of the vector K-subspace of K^m generated by the n rows of the matrix A, in other words,*

$$\dim_K\langle A_1, \ldots, A_n\rangle,$$

which is denoted by $\mathrm{rg}_f(A) = \mathrm{rg}_f A$.

(ii) *The rank of the matrix A by columns is the dimension of the vector K-subspace of K^n generated by the m columns of the matrix A, in other words,*

$$\dim_K\langle A^1, \ldots, A^m\rangle,$$

which is denoted by $\mathrm{rg}_c(A) = \mathrm{rg}_c A$.

Theorem. *Let $A \in M_{n,m}(K)$. Then, the rank of the matrix A by rows and by columns coincides and this is called simply the rank of the matrix A, which is denoted by $\mathrm{rg}\, A$.*

Properties. (i) *If we interchange two rows (two columns) of a matrix A, the rank of the matrix does not change.*

(ii) *If we replace a row or a column of a matrix A by a non-null multiple of it, the rank of the matrix does not change.*

(iii) *If we replace a row or a column of a matrix A by it plus a non-null multiple of any other different row (or column) of the matrix A, the rank of the matrix does not change.*

Basic changes make easier the computation of the rank of a matrix, as it is shown in the following example.

Example. *Let* $A = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & -5 \\ 7 & 8 & 3 \end{pmatrix}$ *be a matrix. Calculate* $\operatorname{rg} A$.

$$\begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & -5 \\ 7 & 8 & 3 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 0 \\ -5 & 1 & 2 \\ 3 & 8 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 2 \\ 3 & 8 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 8 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -9 \end{pmatrix}.$$

Using the definition of the rank of a matrix, it is easy to note that the rank of the last matrix is 3. *Thus* $\operatorname{rg} \begin{pmatrix} 0 & 0 & 2 \\ 2 & 1 & -5 \\ 7 & 8 & 3 \end{pmatrix} = 3$.

2 Systems of linear equations

Definition. *Let K be a field. A system of m linear equations and n indeterminates (variables) over the field K corresponds to:*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where $a_{ij}, b_i \in K, 1 \leq i \leq m, 1 \leq j \leq n$. The matrix form of the previous system of linear equations corresponds to $AX = B$, where

$$A = (a_{ij}) \text{ is the coefficient matrix of the system}$$

$$B = (b_i) \text{ is the column matrix of the constant terms}$$

$$X = (x_j) \text{ is the column matrix of indeterminates (or variables)}$$

$$(A|B) \text{is the expanded matrix of the system.}$$

Definition. *Let $AX = B$ be a system of m linear equations and n indeterminates (variables) over the field K. Then the column matrix* $\alpha = (\alpha_i) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$ *is said to be a solution of the given system of linear equations if $A\alpha = B$ holds.*

Definition. *By definition, if the system of linear equations $AX = B$ does not have any solution, then the system is said to be incompatible, and in the other case, the system is said to be compatible. In the latter case, if the system has an unique solution, then the system is called a determined compatible system and if not, it is called an undetermined compatible system.*

Remark. *The system of linear equations of type $AX = 0$ is called an homogeneous system, which is always compatible, since at least the column matrix 0 is a solution to it.*

Theorem. *(Rouché-Frobenius). Let $AX = B$ be a system of m linear equations and n indeterminates (variables) over the field K. Then, the system $AX = B$ is compatible if and only if $\operatorname{rg}(A) = \operatorname{rg}(A|B)$ holds. (Note that $\operatorname{rg}(A|B)$ can also be expressed by $\operatorname{rg}(\overline{A})$.)*

Proof. *The system $AX = B$ has a solution $\iff \exists \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^t$ such that $A\alpha = B \iff \exists \alpha_1, \ldots, \alpha_n \in K$ such that $\alpha_1 A^1 + \cdots + \alpha_n A^n = B \iff \langle A^1, \ldots, A^n \rangle = \langle A^1, \ldots, A^n, B \rangle \iff \dim\langle A^1, \ldots, A^n \rangle = \dim\langle A^1, \ldots, A^n, B \rangle$.*

Lemma. *Let $A \in M_n(K)$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$ a column matrix. The folowing three statements are equivalent:*

(i) *The system of linear equations $AX = B$ is determined compatible.*

(ii) *The homogeneous linear system $AX = 0$ has an unique solution.*

(iii) *$A \in \mathrm{GL}_n(K)$ i.e the matrix A is invertible.*

Remark. *Let us consider the following system of linear equations:*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

If we make the following (basic) changes in the previous system of linear equations, then the set of solutions of the new system of linear equations does not change.

(i) Interchange any two equations.

(ii) Replace an equation by a non-null multiple of it.

(iii) Replace an equation of the system by it plus a non-null multiple of another equation of the system.

As a consequence, if in the expanded matrix $(A|B)$ the basic changes are made only on its rows, then the associated first and the second system of linear equations have the same solutions.

Example. *Let us consider the following system of linear equations:*

$$x_1 - x_2 + x_3 = 0$$

$$x_2 + x_3 = 1$$

$$x_1 + x_2 = 0$$

The expanded matrix corresponding to the system is: $\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$.

If we make basic changes only on its rows, we get the following matrix: $\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & -2 \end{pmatrix}$.

It is clear that $\mathrm{rg}(A) = \mathrm{rg}(A|B) = 3$. *Thus, the system is determined compatible and the new system associated to the expanded matrix corresponds to:*

$$x_1 - x_2 + x_3 = 0$$

$$x_2 + x_3 = 1$$

$$-3x_3 = -2.$$

So, the unique solution is $x_3 = 2/3$, $x_2 = 1/3$ *and* $x_1 = -1/3$.

3 The symmetric group

Definition. *Let* $X = \{1, \ldots, n\}$ *be a set. The set formed by all the bijective maps from* X *to* X *is denoted by* S_n. *In the set* S_n, *the composition* $(\circ)$ *of maps is an inner operation and* $(S_n, \circ)$ *has a group structure. This mentioned group is called also the symmetric group of order* n, *and the elements of* S_n *are called permutations on* X.

Definition. *Let us consider $\tau \in S_n$. By definition, τ is called a transposition or a 2-cycle if there exist different $i, j \in \{1, \dots, n\}$ such that $\tau(i) = j$ and $\tau(j) = i$ and the images by τ of the rest of indices are fixed. In a short way, $\tau = (ij)$ is written.*

Theorem. *Any permutation of S_n can be written as the product of transpositions, being the product the composition of maps.*

Theorem. *Let us consider $\sigma \in S_n$. If σ admits two decompositions as product of transpositions, then the number of components of those decompositions is always even or always odd.*

Definition. *Let us consider $\sigma \in S_n$. We say that the permutation σ is odd (or even) if the number of components in its decomposition as product of transpositions is odd (or even), respectively. The signature of σ, denoted by $\varepsilon(\sigma)$, is defined as follows:*

$$\varepsilon(\sigma) = \begin{cases} 1 & \text{if the permutation } \sigma \text{ is even} \\ -1 & \text{otherwise.} \end{cases}$$

If the decomposition of σ as a product of transpositions corresponds to $\sigma = \tau_1 \cdots \tau_r$, then $\varepsilon(\sigma) = (-1)^r$.

3.1 Determinant of a square matrix

Definition. *Let $A \in M_n(K)$ be a square matrix. The determinant of A, which is denoted by $|A|$ or by $\det A$ is defined as follows,*

$$\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

Note that $\det A$ is an element of K. In addition to this, in each addend

$$\varepsilon(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)},$$

an unique element of each row and each column of the matrix A appears.

Examples. (i) *If $n = 2$ and $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then*

$$\det A = a_{11}a_{22} - a_{12}a_{21}.$$

(ii) *If $n = 3$ and $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then*

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

(iii) $\det I_n = 1$. *Proof: If $\sigma \neq 1$, there exists $i \in X$ such that $\sigma(i) \neq i$. Then $a_{i,\sigma(i)} = 0$, since this element is out from the diagonal of the matrix. Then, for any $\sigma \neq 1$, the term $a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} = 0$, and consequently* $\det I_n = a_{11}a_{22}\cdots a_{nn} = 1$.

(iv) *If* $A = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$ *is a diagonal matrix, then* $\det A = \lambda_1\lambda_2\cdots\lambda_n$.

In the following, we enumerate the basic properties of the determinant of a square matrix.

Theorem. *Let A be a square matrix.*

(i) *If any two rows or any two columns of the matrix A are equal, then* $\det A = 0$.

(ii) *If we interchange in a matrix two rows (or two columns), then the sign of its determinant changes.*

(iii) *If we multiply a row (or a column) of the matrix A by a scalar, then the value of its determinant is also multiplied by this scalar.*

(iv) *If we add to a row of the matrix (or to a column of the matrix) a linear combination of different rows (or different columns), then the value of the determinant of the resulting matrix does not change.*

(v) *The following properties hold,*

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a'_{i1} + a''_{i1} & \dots & a'_{in} + a''_{in} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a'_{i1} & \dots & a'_{in} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a''_{i1} & \dots & a''_{in} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}.$$

(vi) $|A| = |A^t|$.

(vii) $|AB| = |A||B|$.

Definitions. (i) *If we eliminate some rows and some columns of the matrix A, we get a submatrix of A. In other words, if $A \in M_{m,n}(K)$ and $i_1, \dots, i_r \in \{1, \dots, m\}$, $j_1, \dots, j_s \in \{1, \dots, n\}$, then the following matrix is a submatrix of A:*

$$\begin{pmatrix} a_{i_1j_1} & \dots & a_{i_1j_s} \\ \vdots & \vdots & \vdots \\ a_{i_rj_1} & \dots & a_{i_rj_s} \end{pmatrix}.$$

(ii) *The determinant of any square submatrix of the matrix A is called a minor and the order of the minor is the order of its corresponding submatrix.*

(iii) *If $A \in M_n(K)$, the determinant M_{ij} of the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the i-th row and the j-th column of the matrix A is called the minor of element a_{ij}.*

(iv) *The cofactor A_{ij} of the element a_{ij} in* $\det A$ *is defined as,* $A_{ij} = (-1)^{i+j} M_{ij}$.

Proposition. *If $n > 1$ and $A \in M_n(K)$, then the determinant of A can be computed as follows,*

$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij}, \text{ for any } i \in \{1, 2, \ldots, n\} \text{ (the determinant is developed by a row)}$$

or

$$\det A = \sum_{i=1}^{n} a_{ij} A_{ij}, \text{ for any } j \in \{1, 2, \ldots, n\} \text{ (the determinant is developed by a column).}$$

Definition. *Let $A \in M_n(K)$. If* $\det A = 0$*, then the matrix A is called singular and if not, the matrix A is said to be non-singular.*

Theorem. *Let K be a field and $A = (a_{ij})$ be a square matrix whose coefficients are in K. Then A is said to be invertible if and only if* $\det(A)$ *is an element of K which is non-zero, and hence invertible in K. In that case, the inverse matrix of A is a square matrix (denoted by $A^{-1} = (n_{ij})$) whose coefficients n_{ij} are:*

$$n_{ij} = \det(A)^{-1}(-1)^{i+j} \det(A_{ji}),$$

being A_{ji} the matrix coming from A after eliminating the j-th row and the i-th column of it.

Corollary. *A square matrix A of order $n \times n$ is invertible if and only if A is not singular.*

Definition. *The adjoint matrix of the matrix $A \in M_n(K)$ is defined as follows,*

$$adj(A) = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & & A_{nn} \end{pmatrix}.$$

The formula to compute the inverse matrix of an invertible matrix is:

$$A^{-1} = \frac{1}{\det A} (adj(A))^t.$$

Example. *Compute the inverse matrix of* $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$.

Solution: $A^{-1} = \begin{pmatrix} 7/2 & -3 & 1/2 \\ -1/2 & 0 & 1/2 \\ -1/2 & 1 & -1/2 \end{pmatrix}$.

In the following, some properties of inverse matrices are given.

Properties. *Let* $A, B \in M_n(K)$ *be invertible matrices. Then,*

(i) $(A^{-1})^{-1} = A$

(ii) $(AB)^{-1} = B^{-1}A^{-1}$

(iii) $(A^t)^{-1} = (A^{-1})^t$

(iv) $\det(A^{-1}) = 1/\det A$

3.2 Cramer systems

Definition. *A system of linear equations is said to be a Cramer system, if the number of equations coincides with the number of variables and the coefficient matrix associated to the system is not singular. In other words, a system of* n *linear equations with* n *variables of the type,*

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n,$$

such that

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$

and $\det A \neq 0$ *is called a Cramer system.*

Theorem. *A Cramer system of* n *equations and* n *variables is a determined compatible system (in fact,* $\mathrm{rg}(A) = \mathrm{rg}(\overline{A}) = n$*), and the unique solution of the system is computed* *by the following formulas,*

$$x_1 = \frac{1}{\det A}\begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_n & a_{n2} & \vdots & a_{nn} \end{vmatrix}, x_2 = \frac{1}{\det A}\begin{vmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & b_n & \vdots & a_{nn} \end{vmatrix},$$

$$\dots, x_n = \frac{1}{\det A}\begin{vmatrix} a_{11} & a_{12} & \dots & b_1 \\ a_{21} & a_{22} & \dots & b_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \vdots & b_n \end{vmatrix}.$$

Proof. *Note that*

$$\begin{vmatrix} a_{11} & \dots & b_1 & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & b_n & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \dots & a_{11}x_1 + \dots + a_{1n}x_n & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{n1}x_1 + \dots + a_{nn}x_n & \dots & a_{nn} \end{vmatrix}.$$

Using the determinant's properties the last determinant corresponds to:

$$\begin{vmatrix} a_{11} & \dots & a_{11}x_1 & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{n1}x_1 & \dots & a_{nn} \end{vmatrix} + \dots + \begin{vmatrix} a_{11} & \dots & a_{1n}x_n & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn}x_n & \dots & a_{nn} \end{vmatrix}.$$

Note that in all the previous addends the factor x_j ($j \in \{1, \dots, n\}$) can be taken away and that at least for all the addends different to the i-th addend its value is 0, since for them at least two columns are equal. This means that the last expression coincides with $x_i \det A$, and from it the formula for x_i is obtained.

Example. *Compute the following system of linear equations using the Cramer's rule:*

$$x - 3y + 2z = 9$$

$$-2x - 3z = -6$$

$$4x - 10y + 9z = 12.$$

Solution: The matrix form of the previous system of linear equations corresponds to:

$$\begin{pmatrix} 1 & -3 & 2 \\ -2 & 0 & -3 \\ 4 & -10 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \\ 12 \end{pmatrix}.$$

In this case, the system is a Cramer system, since the number of equations and the number of variables are equal (= 3) and $\det A = -8 \neq 0$*. Then, the unique solution is given by*

$$x = \frac{1}{-8}\begin{vmatrix} 9 & -3 & 2 \\ -6 & 0 & -3 \\ 12 & -10 & 9 \end{vmatrix} = \frac{51}{2}, y = \frac{1}{-8}\begin{vmatrix} 1 & 9 & 2 \\ -2 & -6 & -3 \\ 4 & -12 & 9 \end{vmatrix} = -\frac{9}{2},$$

$$z = \frac{1}{-8}\begin{vmatrix} 1 & -3 & 9 \\ -2 & 0 & -6 \\ 4 & -10 & 12 \end{vmatrix} = -15.$$

3.3 Rank of a matrix using determinants

Theorem. *The rank of any* A *matrix is the maximum order of its non-null minors. In other words,* $\mathrm{rg}(A) = r$ *if and only if there exists a non-null minor of order* r *and all the minors of order* $r+1$ *are nulls.*

Example. *Let* $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -2 & 1 & 1 \\ 4 & 3 & 6 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ *be a matrix. Compute the rank of* A*.*

There exists an unique minor of order 4*. Precisely,* $\begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -2 & 1 & 1 \\ 4 & 3 & 6 & -1 \\ 0 & 0 & 0 & 2 \end{vmatrix}$*, and its value is equal to* 0 *(developing it by the* 4*-th row). On the other hand, the minor of order* 3 *obtained by the determinant of the submatrix of the matrix* A *formed by the rows* $1, 2$ *and* 4 *and the columns* $1, 2$ *and* 4 *is different from* 0 *(=* 2*). Thus, the rank of the matrix* A *is* 3*.*