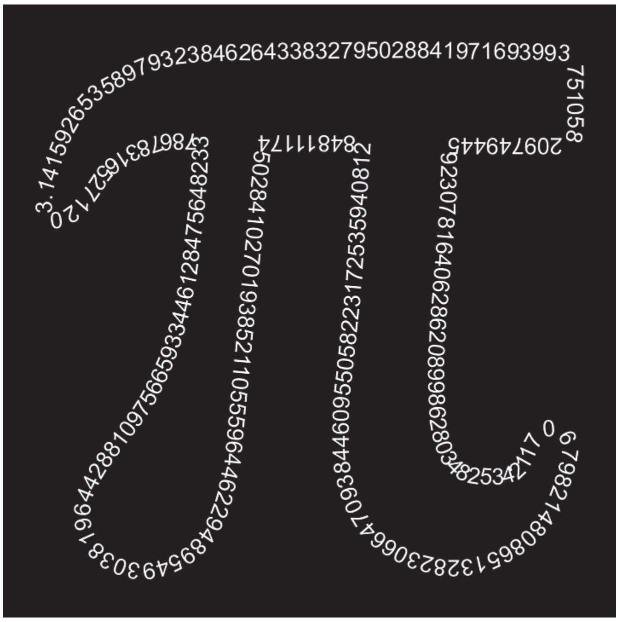
MATHS BASIC COURSE FOR UNDERGRADUATES



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7th SUBJECT: POLYNOMIAL INEQUATIONS
Definitions. Polynomial inequations. Some classical inequalites.

1 Definitions

Definition. *In mathematical language, a relation between at least two elements that uses at least one of the following signs is called inequality:*

$$> \text{ bigger than}$$

$$< \text{ smaller than}$$

$$\geq \text{ bigger than or equal to}$$

$$\leq \text{ smaller than or equal to.}$$

Examples. *The following expressions are inequalities, and they are read as follows:*

(i) $6 > 4$*:* 6 *is bigger than* 4

(ii) $a > b$*:* a *is bigger than* b

(iii) $x < -2$*: the variable* x *takes values smaller than* -2*, but not the* -2 *value*

(iv) $2 < x < 7$*: the variable* x *takes values between* 2 *and* 7*, but without taking the limit values* 2 *and* 7

(v) $-1 \leq y < 3$*: the variable* y *takes values between* -1 *and* 3
taking -1 *as the lowest value, and leaving the highest limit,* 3*, out .*

(vi) $z^2 \geq 9$*: the variable* z *takes the values for which the square is equal or bigger than* 9*.*

Properties. *There exist the following basic rules related to the order of real numbers:*

(i) *If* $x \in \mathbb{R}$*, one of the following three conditions is fulfilled:* $x > 0$*,* $x < 0$ *or* $x = 0$*.*

(ii) *If* $x > y$*, then* $-x < -y$*.*

(iii) *If* $x > y$ *and* $c \in \mathbb{R}$*, then* $x + c > y + c$*.*

(iv) *If* $x > 0$ *and* $y > 0$*, then* $xy > 0$*.*

(v) *If* $x > y$ *and* $y > z$*, then* $x > z$*.*

2 Polynomial inequations

Definition. *If there are one or more variables in an inequality and if this inequality is only fulfilled for some values of those variables, then that inequality is called inequation. In another words, an inequation is an inequality between two algebraic expressions, and its result is the subset of real numbers that fulfill that inequality.*

Examples. *The following expressions are inequations that have more than one variable.*

(i) $x + y < 3$*: "*$x + y$*" is smaller than* 3

(ii) $y - x + 5 \geq 0$*: "*$y - x + 5$*" is bigger than or equal to* 0

(iii) $x + 2y - z \leq 4$*: "*$x + 2y - z$*" is smaller than or equal to* 4

(iv) $x^2 + y^2 \leq 25$*: "*$x^2 + y^2$*" is smaller than or equal to* 25*.*

The first three inequations are linear, while on the other hand, the last one is not.

Remark. *Graphically, in order to represent inequalities or inequations, we can use the real number line, or depending on the number of variables, a cartesian coordinate diagram of dimension of order* 2 *or* 3*.*

Examples. *These are some examples (exercises) about inequations.*

(i) *If* $a, b \in \mathbb{R}$ *and* $b > 0$*, solve the inequation* $|x - a| \leq b$*.*

(ii) *Find which values of* x *fulfill the inequality* $|x - 3| < 2|x + 3|$*.*

Properties. *Two inequations are considered equivalent if they have the same result. Equivalent inequations can be achieved if:*

(i) *We add or subtract the same number on both sides of the inequality.*

(ii) *Multiply or divide by the same positive number on both sides.*

(iii) *Multiply or divide by the same negative number on both sides and then, change the direction of the inequality.*

Definition. *By definition, an inequation is called linear or of first degree if on each side there are polynomials of order one. To solve them, we follow a process that is similar to solving first degree equations. On the other hand, second degree inequations are those that have a polynomial of degree two in at least one of its sides, and have no polynomials of degree three or bigger than three. To solve them, we follow the next steps:*

(i) *We operate until we obtain an equivalent inequation that, on one of its sides is* 0*.*

(ii) *The polynomial left on the other side is factorized, this is, we express it in the form:* $a(x - r_1)(x - r_2)$.

(iii) *We calculate the signs of the three regions of the real number line delimited by the previous two roots, and then we determine the sign of the polynomial in those regions.*

(iv) *We write the solution, including or excluding the roots according to the type of the inequality.*

Remark. *In the sides of a polynomial inequation, there can appear polynomials of degree bigger than* 2. *To solve them, we follow a method that is similar to the one used for inequations of second degree.*

(i) *Suppose that the polynomial found in the inequation is of the following type:*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \sim 0,$$

where $\sim$ *can be:* $<, >, \leq$, *or* $\geq$.

(ii) *We calculate the roots of the equation* $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$. *We could obtain at most* n *real roots; lets call them* $\alpha_1, \dots \alpha_n$.

(iii) *We place these roots in a table and then we study the sign in each region.*

Example. *Solve the following inequality:* $x^3 + x \leq 4x^2 - 6$.

Proof. *The previous inequality is equivalent to:* $x^3 + x - 4x^2 + 6 \leq 0$, *and the third degree polynomial can be factorized as* $x^3 + x - 4x^2 + 6 = (x+1)(x-2)(x-3)$. *Therefore, the initial inequality is equivalent to:* $(x+1)(x-2)(x-3) \leq 0$. *Now, we analyze the signs by using the following table:*

	$(-\infty, -1)$	$(-1, 2)$	$(2, 3)$	$(3, \infty)$
$x+1$	$-$	$+$	$+$	$+$
$x-2$	$-$	$-$	$+$	$+$
$x-3$	$-$	$-$	$-$	$+$
$(x+1)(x-2)(x-3)$	$-$	$+$	$-$	$+$

Hence, $x^3 + x \leq 4x^2 - 6$ *if and only if* $x \in (-\infty, -1] \cup [2, 3]$.

3 Some classical inequalities

Lemma. Cauchy-Schwartz inequality. *Let* $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n$ *be two families of real numbers. The following is fulfilled:*

$$\sum_{i=1}^n (a_i b_i) \leq \sqrt{\sum_{i=1}^n a_i^2} \sqrt{\sum_{i=1}^n b_i^2}.$$

The equality holds if and only if there exists some λ for which $a_i = \lambda b_i$, for any i.

Proof. *Suppose that $\sum_{i=1}^{n} a_i^2 \neq 0 \neq \sum_{i=1}^{n} b_i^2$. In other cases, for any i, $a_i = 0 = b_i$, and the inequality is obvious. Let $\alpha, \beta \in \mathbb{R}$. Then:*

$$0 \leq \sum_{i=1}^{n} (\alpha a_i - \beta b_i)^2 = \sum_{i=1}^{n} \left(\alpha^2 a_i^2 + \beta^2 b_i^2 - 2\alpha\beta a_i b_i\right),$$

this is, $2\alpha\beta\sum_{i=1}^{n} a_i b_i \leq \alpha^2\sum_{i=1}^{n} a_i^2 + \beta^2\sum_{i=1}^{n} b_i^2$. Now, by taking $\alpha = \sqrt{\sum_{i=1}^{n} b_i^2}$ and $\beta = \sqrt{\sum_{i=1}^{n} a_i^2}$, the result is proved.

Lemma. Minkowski inequality. *In the conditions of the previous lemma, the following is fulfilled:*

$$\sqrt{\sum_{i=1}^{n}(a_i + b_i)^2} \leq \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}$$

Proof. *What we want to prove is equivalent to*

$$\sum_{i=1}^{n}(a_i + b_i)^2 \leq \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2},$$

this is, summarizing, $\sum_{i=1}^{n}(a_i b_i) \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$, and this is the result of the previous lemma.

Lemma. Triangle inequality. *Let $\{a_i\}_{i=1}^{n}$ be a family of real numbers. Then,*

$$|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|.$$

Proof. *The previous inequality is called the triangle inequality, since for the case $n = 2$, if we address the sides of the triangle as a_1 and a_2, this inequality indicates that one side cannot be longer than the sum of the other two. Since $-|a_i| \leq a_i$, by adding them, $-(|a_1| + |a_2| + \cdots + |a_n|) \leq a_1 + a_2 + \cdots + a_n \leq |a_1| + |a_2| + \cdots + |a_n|$.*

Lemma. *Let $\{a_i\}_{i=1}^{n}, \{b_i > 0\}_{i=1}^{n}$ be two families of real numbers. If*

$$\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq \cdots \leq \frac{a_n}{b_n} \quad \Rightarrow \quad \frac{a_1}{b_1} \leq \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq \frac{a_n}{b_n}.$$

Proof. *Let us call* $\alpha_i = \frac{a_i}{b_i}$. *Then,* $\alpha_1 \leq \cdots \leq \alpha_n$. *Thus,*

$$\alpha_1 b_1 \leq \alpha_1 b_1 \leq \alpha_n b_1$$

$$\alpha_1 b_2 \leq \alpha_2 b_2 \leq \alpha_n b_2$$

$$\cdots$$

$$\alpha_1 b_n \leq \alpha_n b_n \leq \alpha_n b_n$$

Therefore,

$$\alpha_1(b_1 + \cdots + b_n) \leq (a_1 + \cdots + a_n) \leq \alpha_n(b_1 + \cdots + b_n),$$

and dividing by $b_1 + \cdots + b_n$, *we obtain the result. Specifically, if* $b_1 = \cdots = b_n = 1$, *the arithmetic mean of any family of numbers is between the lowest and the biggest numbers of that family.*

Lemma. Bernoulli's inequality. *If* h *is a real number bigger than -1 and* n *is an integer, then* $(1+h)^n \geq 1 + nh$.

Proof. *The proof is done by induction on* n. *If* $n = 1$, *then the result is immediate. Suppose now that the result is fulfilled for* n, *and let us see the case for* $n+1$. $(1+h)^n \geq 1+nh$, *and since* $1+h > 0$, *multiplying by it, we obtain* $(1+h)^{n+1} \geq (1+nh)(1+h) = 1+(n+1)h+nh^2 \geq 1+(n+1)h$. *If* $h > 0$, *instead of applying induction, it is enough to use the binomial expansion:*

$$(1+h)^n = \binom{n}{0}h^n + \binom{n}{1}h^{n-1} + \binom{n}{2}h^{n-2} + \cdots + \binom{n}{n-1}h + \binom{n}{n}$$

$$\geq \binom{n}{n-1}h + \binom{n}{n} = nh + 1.$$

Lemma. *Inequality of arithmetic and geometric means. If* $\{a_i\}_{i=1}^n$ *is a family of positive real numbers, then*

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n} \geq \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}.$$