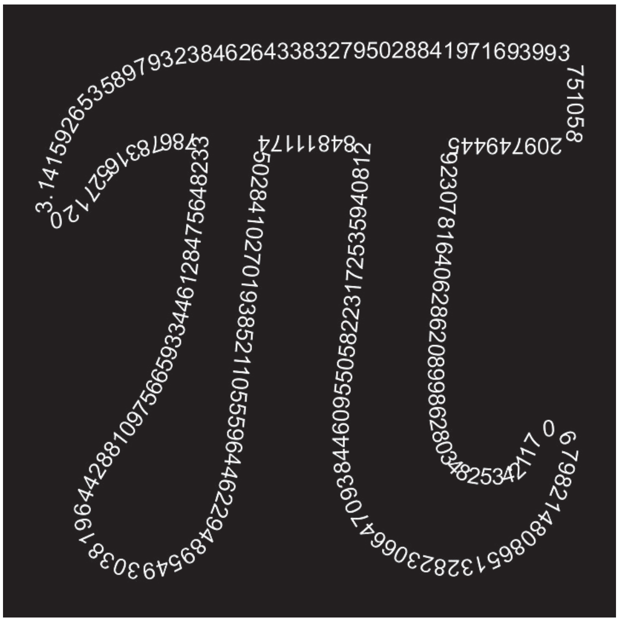
MATHS BASIC COURSE FOR UNDERGRADUATES



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3rd SUBJECT: FUNCTIONS

1 Functions. Examples

Definition. *Let A and B be two sets. A rule from the set A to the set B sending any $a \in A$ to an unique element of B is called a function from A to B, and it is denoted usually by f (or g, h...). That element of B is called the image by f of the element $a \in A$ and it is denoted by $f(a)$. In addition to this,*

(i) *A is said to be the domain of f, and it is expressed by $A = D(f)$.*

(ii) *B is said to be the codomain of f and the set of values that the function f takes on as output is termed the image of the function f, which is sometimes also referred to as the range of the function f, and which is a subset of B.*

Remark. The function f from A to B can be expressed by $f : A \longrightarrow B$ or by $A \xrightarrow{f} B$.

Examples. (i) *$f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = x^2$ or $f(x) = \sin x$ are both functions from $\mathbb{R}$ to $\mathbb{R}$.*

(ii) *$f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = 1/x$ is not a function, since the element 0 does not have any image in $\mathbb{R}$. However, if we replace the domain $\mathbb{R}$ of f by $\mathbb{R} - \{0\}$, then f becomes a function.*

(iii) *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an expression with the following rule,*

$$f(x) = \begin{cases} x + 1, & \text{if } x > 0 \\ 1, & \text{if } x \leq 0. \end{cases}$$

Then f is a function and it is said that f is defined by parts.

Remind that if $A, B \subseteq \mathbb{R}$, then the function $f : A \longrightarrow B$ can be addressed in the plane, marking for all $x \in A$ the points $(x, f(x))$.

(iv) *Let A and B be two sets and b_0 a fixed element of B. If for any $a \in A$, the function f maps $f(a) = b_0$, then f is a function, which is called a constant function.*

(v) *The identity function can be defined on any set A, as follows: $1_A : A \longrightarrow A$ such that $1_A(x) = x$, for any $x \in A$.*

Given a function $f : A \longrightarrow B$, sometimes we could be interested in restricting it on some subset of the domain A. This the reason why the following definition is introduced.

Definition. *Let $f : A \longrightarrow B$ be a function and $S \subseteq A$. The restriction of f on the subset S, which is denoted by $f_{|S}$, is the function obtained restricting the domain of f to the subset S. Thus the function $f_{|S}$ satisfies the following two properties:*

(i) $f_{|S} : S \longrightarrow B$ *(the domain is restricted to S)*

(ii) $f_{|S}(s) = f(s), \forall s \in S$ *(for any element of S its image by $f_{|S}$ coincides with its image by f).*

2 Types of functions

Definition. *Let $f : A \longrightarrow B$ be a function.*

(i) *f is said to be injective, if any two (different) elements of A have different image in B.*

(ii) *f is said to be surjective if all the elements of B are image of some element of A.*

(iii) *f is said to be bijective if f is injective and surjective.*

Examples. (i) *Let $f : A \longrightarrow B$ be a constant function. Then, in order to be f a injective function $|A|$ must be equal to 1, and in order to be f a surjective function $|B|$ must be equal to 1. Thus, f is a bijective function if and only if $|A| = |B| = 1$.*

(ii) *It is obvious that the identity function 1_A is always bijective.*

(iii) *There is no bijective function from the set $A = \{a, b, c\}$ to the set $B = \{x, y\}$, and neither an injective function between those sets. However, we can build surjective functions from A to B.*

Proposition. *Let A and B be two sets. Then,*

(i) *there exists an injective function $f : A \longrightarrow B$ if and only if $|A| \leq |B|$;*

(ii) *there exists a surjective function $f : A \longrightarrow B$ if and only if $|A| \geq |B|$;*

(iii) *there exists a bijective function $f : A \longrightarrow B$ if and only if $|A| = |B|$.*

Sometimes, when the function is simple, we can represent it using the Venn's diagrams, and in that case it is pretty easy to analyze which type of function it is. If not, the formal way to check which type of function $f : A \longrightarrow B$ is, could be to follow the next method:

(i) f injective: take any two elements of A that have the same image in B and prove that

$$a, a' \in A \text{ and } f(a) = f(a') \Longrightarrow a = a'$$

(ii) f surjective: prove it directly by using the definition.

$$\forall b \in B, \exists a \in A \mid f(a) = b$$

or equivalently,

$$b \in B \Longrightarrow \exists a \in A \mid f(a) = b$$

(iii) f bijective: check if f is injective and surjective at the same time.

Example. *The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = 2x + 3$ for any $x \in \mathbb{R}$, is a bijective function.*

Definition. *Let $f : A \longrightarrow B$ be a function.*

(i) *If $S \subseteq A$, the image of S by f (or the direct image of S by f), which is denoted by $f(S)$, is the set formed by the image of all the elements of S by f. In other words,*

$$f(S) = \{f(s) \mid s \in S\}.$$

In particular, the subset of B, $f(A)$, is the set formed by all the images of elements of A, which is called the image of f and is denoted by Imf.

(ii) *If $T \subseteq B$, the inverse image of T by f, which is denoted by $f^{-1}(T)$, is the set formed by those elements of A whose images by f are in the subset T. It means,*

$$f^{-1}(T) = \{a \in A \mid f(a) \in T\}.$$

Remark. If the subset T of B is formed by an unique element b ($T = \{b\}$), the inverse image of T by f is denoted by $f^{-1}(b)$ instead of $f^{-1}(\{b\})$.

Example. *Let $f(x) = x^2$ be a function from $\mathbb{R}$ to $\mathbb{R}$. The following images and inverse images are computed: $imf = [0, +\infty)$, $f([0, 2]) = [0, 4]$, $f([2, +\infty)) = [4, +\infty)$, $f((-\infty, -1) \cup [2, +\infty)) = (1, +\infty)$, $f^{-1}(1) = \{-1, 1\}$, $f^{-1}(-1) = \emptyset$, $f^{-1}([-1, 0]) = \{0\}$ and $f^{-1}((1, +\infty)) = (-\infty, -1) \cup (1, +\infty)$.*

Remark. *Remind now the notation using with intervals. If $a, b \in \mathbb{R}$, then $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$, $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$, $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$ and $[a, +\infty) = \{x \in \mathbb{R} \mid a \leq x\}$. In an analogous way, all the possible combinations among them can be defined.*

Suppose now that a function $f : A \longrightarrow B$ can be represented by the Venn's diagrams, and that we are able to express all the arrows involved in it backwards, i.e. instead of mapping from A to B, mapping from B to A. The following question arrives: is the obtained rule a function? The answer could be affirmative or even negative.

3 Inverse function

According to the previous question the new rule would be a function if and only if there is an unique arrow arriving to any element of B, which is equivalent to saying that the function f is bijective. Summarizing, given a function $f : A \longrightarrow B$, **the rule obtained representing the arrows backwards is again a function if and only if f is bijective**. In that case, the new function is called the inverse function of f, and it is represented by f^{-1}. Thus, $f^{-1} : B \longrightarrow A$.
Remind that the meaning of an arrow from an element $a \in A$ to an element $b \in B$ is quite simple: $f(a) = b$. When it happens, the expression f^{-1} sends also an arrow from b to a, which is expressed by $f^{-1}(b) = a$. Now keeping in mind this remark, we introduce the more formal definition of the inverse function, as follows:

Definition. *Let $f : A \longrightarrow B$ be a bijective function. Then the inverse function of f is defined by means of the next two properties:*

(i) *f^{-1} maps from B to A, i.e. $f^{-1} : B \longrightarrow A$*

(ii) *$f^{-1}(b) = a$ if and only if $f(a) = b$.*

Examples. *1) Let $f : \mathbb{R} \longrightarrow [0, +\infty)$ be a function such that $f(x) = x^2, \forall x \in \mathbb{R}$. It is obvious that f is not bijective (it is surjective but it is not injective). As a consequence, the function f^{-1} does not exist.*

2) Let $g : [0, +\infty) \longrightarrow [0, \infty)$ be a function such that $g(x) = x^2$, $\forall x \in [0, +\infty)$. It holds that g is a restriction of the function f defined in the previous example to the interval $[0, \infty)$, and g is bijective. Therefore, there exists the inverse function of g, which is denoted by $g^{-1} : [0, +\infty) \longrightarrow [0, +\infty)$. Now we compute the rule for this new function. Keep in mind that $g^{-1}(y) = x$ if and only if $g(x) = y$. Thus, to compute the image by g^{-1} of any element $y \in [0, +\infty)$, we should solve the equation $g(x) = y$ being x an unknown value. We get that $x = \sqrt{y}$, and it is the same, as what was done as checking if the function g is surjective or not. In conclusion, $g^{-1}(y) = \sqrt{y}$ or renaming the variable, the rule $g^{-1}(x) = \sqrt{x}$ is obtained, for any $x \in [0, +\infty)$.

3) Let $h : (-\infty, 0] \longrightarrow [0, \infty)$ be a function such that $h(x) = x^2$. Clearly h is the restriction of the function f defined in the example 1 to the interval $(-\infty, 0]$. It is clear that h is bijective, and working in an analogous way as in the example 2, we get that the inverse function of h corresponds to $h^{-1}(x) = -\sqrt{x}$, for any $x \in [0, -\infty)$.

Remark. *Let $f : A \longrightarrow B$ be a bijective function. If $T \subseteq B$, then the symbol $f^{-1}(T)$ can represent two different concepts:*

(i) *The inverse image of T by the function f. According to this definition, $a \in f^{-1}(T)$ if and only if $f(a) \in T$.*

(ii) *The direct image of T by the function f^{-1} (which exists since f is bijective). According to this second definition,*

$$a \in f^{-1}(T) \iff \exists b \in T \mid a = f^{-1}(b) \text{ (definition of a direct image)}$$

$$\iff \exists b \in T \mid f(a) = b(\text{ by the definition of } f^{-1})$$

$$\iff f(a) \in T.$$

In both cases, the set $f^{-1}(T)$ is formed by the same elements, and consequently there is no ambiguity in using the symbol $f^{-1}(T)$.

On the other hand, in the case the function $f : A \longrightarrow B$ is not bijective, the symbol $f^{-1}(T)$ does make sense in terms of the definition (i), but it does not make sense in terms of the definition using in (ii). If the function f is not bijective, the function f^{-1} does not exist, and consequently we cannot compute the direct image of f^{-1}.

Proposition. *Let $f : A \longrightarrow B$ be a function. If the inverse function of f exists, then it is unique and it is represented by f^{-1}.*

Proposition. *Let $f : A \longrightarrow B$ be a function. If A and B are two finite sets, satisfying $|A| = |B|$,*

(i) *if f is injective, then f is bijective as well;*

(ii) *if f is surjective, then f is bijective as well.*

The previous result does not hold if the sets involved are infinite sets.

4 Composition of functions

Now we introduce an important operation among functions.

Definition. *Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two functions. The composition function among f and g, which is denoted by $g \circ f$ is a function satisfying the following two propertties:*

(i) $g \circ f : A \longrightarrow C$.

(ii) $(g \circ f)(a) = g(f(a)), \forall a \in A$.

In terms of the given definition, in order to compute $g \circ f$, first f is computed and after that g.

Not always it is possible to compose two functions. In the definition of the composition of the functions f and g, the codomain of f and the domain of g are equal; in other words, im $f =$ dom g. In addition to this, it would be possible also compose the previous two functions, even in the case im $f \subseteq$ dom g.

Example. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be two functions, given by the rules $f(x) = x^2$ and $g(x) = x + 2$, respectively. It is possible to compute the following four functions: $f \circ f, f \circ g, g \circ f$ and $g \circ g$, which are again functions from $\mathbb{R}$ to $\mathbb{R}$. The obtained rules are, as follows:*

$$(f \circ f)(x) = f(f(x)) = f(x^2) = \left(x^2\right)^2 = x^4,$$
$$(f \circ g)(x) = f(g(x)) = f(x + 2) = (x + 2)^2 = x^2 + 4x + 4,$$
$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 2,$$
$$(g \circ g)(x) = g(g(x)) = g(x + 2) = (x + 2) + 2 = x + 4.$$

It is clear that in general $f \circ g \neq g \circ f$. Therefore, it is important the order in which the functions are composed in a composition function.

The next theorem is immediate.

Theorem. *Let $f : A \longrightarrow B$ be a function. Then,*

(i) *$f \circ 1_A = f$ and $1_B \circ f = f$. As a consequence, the identity function does not change the behaviour of the initial function in a composition. Note that, on the one hand the identity function 1_A is considered, and on the other hand, the identity function 1_B.*

(ii) *If f is bijective, $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$. In other words, the composition among a bijective function and its inverse function is the corresponding identity function.*