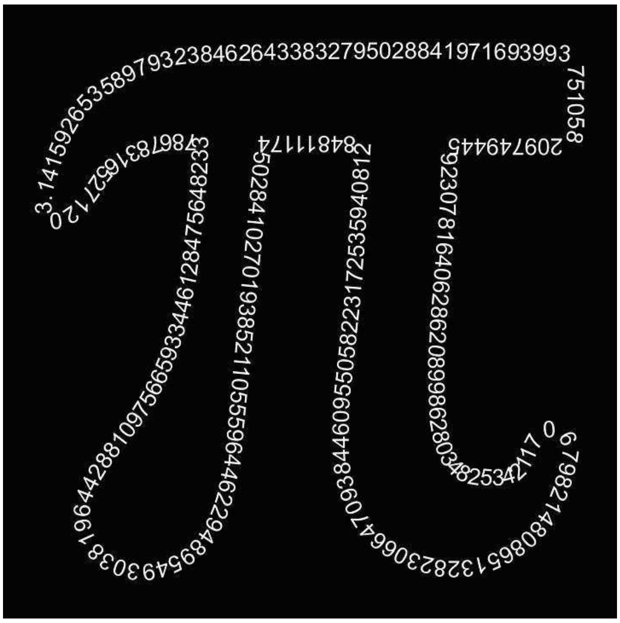
MATHS BASIC COURSE FOR UNDERGRADUATES



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2nd SUBJECT: COMPLEX NUMBERS

Operations. Conjugate. Polar form. Solutions of polynomial equations. Fundamental theorem of algebra.

It is known that the equation $x^2 + 1 = 0$ has no real roots. If we denote by i the number fulfilling $i^2 = -1$ relationship, by definition we call *complex number* to the expression $z = a + ib$, where $a, b \in \mathbb{R}$, being $a = \mathrm{Re}(z)$ the *real part* of the complex number z, and $b = \mathrm{Im}(z)$ the *imaginary part* of the complex number z.

It can be noticed that the real number $a \in \mathbb{R}$ can be identified by the complex number $a = a + i0$. Thus, it can be easily figured out that $\mathbb{R} \subseteq \mathbb{C}$. Sometimes, the complex number $z = a + ib$ (this mode is called the *binomial form of the complex number*), can be identified with the pair $(a, b) \in \mathbb{R}^2$, and therefore we can speak about the *complex plane*.

1 Operations

Let $a, b, c, d \in \mathbb{R}$.

Addition

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

Subtraction

$$(a + ib) - (c + id) = (a - c) + i(b - d)$$

Product

Since $i^2 = -1$, the following can be concluded

$$(a + ib).(c + id) = (ac - bd) + i(ad + bc)$$

Division

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}$$

in the case $c + id \neq 0$

2 Conjugate

If $z = a + ib$ is a complex number, the number $\overline{z} = a - ib$ is called its *conjugate*. In $\mathbb{R}^2$, if we denote as $|z|$ the distance between the origin and the point (a, b), then by applying

Pythagoras' Theorem, we obtain $|z| = \sqrt{a^2 + b^2}$. Also notice that $z.\overline{z} = |z|^2$.

3 Polar form

If $z = a + ib$, the angle θ formed by the $0X$ axis and the line passing through the origin and z is called the *argument* of the complex number z, which is denoted by $arg(a + ib)$. Therefore, by using basic trigonometrical concepts, $a = r\cos\theta$, $b = r\sin\theta$ and we obtain $\arg(a + ib) = \text{arctag}\frac{b}{a}$, being $r = |z|$.
The complex number z can also be addressed as,

$$z = r\left(\cos\theta + i\sin\theta\right).$$

Notice that for the argument $\arg(z)$, there is only one value between $-\pi < \theta \le \pi$, which is called the *principal value*. In addition, we call the *polar form* of the complex number $z = a + bi \neq 0$ to the expression r_θ, where $r = |z|$ is the module of $z = a + bi$, and θ is the argument of $z = a + ib$.

Example. $3i = 3_{\pi/2}; 1 + i = (\sqrt{2})_{\pi/4}$ *and* $-1 - i = (\sqrt{2})_{5\pi/4}$.

Theorem. *(**Moivre.**) If* $z_1 = r_1\left(\cos\theta_1 + i\sin\theta_1\right)$ *and* $z_2 = r_2\left(\cos\theta_2 + i\sin\theta_2\right)$*, then* $z_1 z_2 = r_1 r_2\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$.

Proof.

$$z_1 z_2 = r_1 r_2\left(\cos\theta_1 + i\sin\theta_1\right)\left(\cos\theta_2 + i\sin\theta_2\right) =$$
$$= r_1 r_2\left((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\right) =$$
$$= r_1 r_2\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right).$$

Corollary. *If* $z = r\left(\cos\theta + i\sin\theta\right)$*, then* $z^n = r^n\left(\cos n\theta + i\sin n\theta\right)$*, and* $z^{-n} = r^{-n}\left(\cos n\theta - i\sin n\theta\right)$.

Proof. *It is enough to notice that*

$$z^{-1} = \frac{1}{z} = \frac{1}{r\left(\cos\theta + i\sin\theta\right)} = \frac{\cos\theta - i\sin\theta}{r}.$$

Example. $(2_{\pi/3})^3 = 8_\pi$, $\frac{2_{\pi/3}}{\sqrt{3}_{\pi/4}} = (\frac{2}{\sqrt{3}})_{\pi/3-\pi/4} = (\frac{2}{\sqrt{3}})_{\pi/12}$.

Example. *Calculate* $(-\sqrt{3} + i)^7$*. Notice that the complex number* $z = -\sqrt{3} + i$ *can also be written as* $z = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$*. Thus,*

$$z^7 = 2^7(\cos\frac{35\pi}{6} + i\sin\frac{35\pi}{6}) = 2^7(\cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6}) = 2^7(\frac{\sqrt{3}}{2} + i\frac{-1}{2}) = 2^6(\sqrt{3} - i).$$

Example. *Find the formula for* $\cos(3\theta)$ *depending on* $\cos\theta$*.*

First, we have that $\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$*. Then, by raising it to the third power and by matching the real parts and the imaginary parts, we conclude that* $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$*.*

4 Solutions of polynomial equations

For ease of use, complex numbers are represented by the exponential notation, which is $e^{i\theta} = \cos\theta + i\sin\theta$. With this notation, if $z = re^{i\theta}$, then $\overline{z} = re^{-i\theta}$. Besides, $z_1 = r_1 e^{i\theta_1} = r_2 e^{i\theta_2} = z_2$ if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2k\pi$, being $k \in \mathbb{Z}$.

For example, if we want to solve the equation $z^3 = 1$, we can use the following decomposition:

$$0 = z^3 - 1 = (z-1)(z^2 + z + 1),$$

where 1, $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$ are the roots, also written as 1, $e^{i\frac{2\pi}{3}}$ and $e^{i\frac{4\pi}{3}}$, and which are called *the cube roots of unity*.

In general, if a complex number z fulfills the equation $z^n = 1$, this number is called an *n-th root of unity*.

Proposition. *If $n \in \mathbb{N}$ and $w = e^{i\frac{2\pi}{n}}$, then the n-th roots of unity are $1, w, w^2, \ldots,$ and w^{n-1}.*

Proof. *Let $z = re^{i\theta}$ be one n-th root of unity. Then, $z^n = r^n e^{in\theta} = 1$, and we infer that $r = 1$ and $n\theta = 2k\pi$ (being $k \in \mathbb{Z}$). Thus, $\theta = \frac{2k\pi}{n}$ and $z = e^{i\frac{2k\pi}{n}} = w^k$. Therefore, each n-th root of unity is a power of w. Viceversa, each power of w is an n-th root of unity, because $(w^k)^n = w^{kn} = (e^{i\frac{2\pi}{n}})^{kn} = (e^{i2\pi})^k = 1$.*

Example. *The 4-th roots of unity are $1, e^{i\frac{\pi}{2}}, e^{i\pi}$ and $e^{i\frac{3\pi}{2}}$, also written as $1, i, -1, -i$. The 6-th roots of unity are $1, e^{i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}}, -1, e^{i\frac{4\pi}{3}}$ and $e^{i\frac{5\pi}{3}}$, and they are the vertices of a regular hexagon inscribed in a circle.*

On the other hand, we can use the n-th roots of unity to calculate the n-th roots of any complex number.

Example. *Solve the equation $z^5 = -\sqrt{3} + i$. This is, calculate the fifth roots of the complex number $-\sqrt{3} + i$.*

To do this, use the notation $z_0 = -\sqrt{3} + i = 2e^{i\frac{5\pi}{6}}$. Thus, $\alpha = 2^{\frac{1}{5}} e^{i\frac{\pi}{6}}$ can be a fifth root of z_0. If w is a fifth root of unity, then $(\alpha.w)^5 = \alpha^5 w^5 = \alpha^5 = z_0$. Therefore, $\alpha.w$ is also a fifth root of z_0. Consequently, the fifth roots of $z_0 = -\sqrt{3} + i$ are:

$$\alpha, \alpha e^{i\frac{2\pi}{5}}, \alpha e^{i\frac{4\pi}{5}}, \alpha e^{i\frac{6\pi}{5}}, \alpha e^{i\frac{8\pi}{5}}.$$

In fact, there are five roots which are roots of z_0. If β is another fifth root of z_0, then $\beta^5 = \alpha^5 = z_0$, and from here we obtain $\frac{\beta^5}{\alpha^5} = 1$, and this means that $\frac{\beta}{\alpha} = w$ is a fifth root of unity, and that the expression $\beta = \alpha w$ is in the previous list. As a consequence, the fifth roots of $z_0 = (-\sqrt{3} + i)$ are:

$$2^{\frac{1}{5}} e^{i\frac{\pi}{6}}, 2^{\frac{1}{5}} e^{i\frac{17\pi}{30}}, 2^{\frac{1}{5}} e^{i\frac{29\pi}{30}}, 2^{\frac{1}{5}} e^{i\frac{41\pi}{30}}, 2^{\frac{1}{5}} e^{i\frac{53\pi}{30}}.$$

In general, the previous method indicates that: if β is an n-th root of a complex number, the other n-th roots have the form $\beta w, \beta w^2, \ldots, \beta w^{n-1}$, where $w = e^{i\frac{2\pi}{n}}$.

5 Fundamental theorem of algebra

A polynomial equation has the form $p(x) = 0$, where $p(x)$ is a polynomial with complex coefficients, such as

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, a_i \in \mathbb{C}.$$

Theorem. *Any polynomial equation of degree at least 1, has one root in $\mathbb{C}$. (The proof is not given.)*

Thus, if $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, by the Fundamental theorem of algebra, $p(x)$ has at least one root in $\mathbb{C}$. Let us take for example α_1. Then,

$$p(x) = (x - \alpha_1)p_1(x),$$

where $p_1(x) \in \mathbb{C}[x]$ is a polynomial of degree $n - 1$. By applying again the previous theorem to $p_1(x)$, this one has another root in $\mathbb{C}$, which is denoted by α_2. Therefore, it exists a polynomial $p_2(x) \in \mathbb{C}[x]$ of degree $n-2$ that gives the following decomposition of $p(x)$:

$$p(x) = (x - \alpha_1)(x - \alpha_2)p_2(x).$$

By repeating this reasoning, we can obtain polynomials of degree 1 for which $p(x)$ has the following decomposition:

$$p(x) = a_n(x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_n).$$

Corollary. *Any polynomial of degree n can be factorized in linear polynomials, and in particular, it has n roots in $\mathbb{C}$ (some of them can be repeated).*

Now, if we consider the polynomial equation $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with real coefficients, this is, $a_i \in \mathbb{R}$, then its roots do not necessarily need to be real. However, we can conclude some interesting properties about those roots.

Fact. If $\alpha \in \mathbb{C}$ is a root of the equation $p(x) = 0$, being $p(x) \in \mathbb{R}[x]$, then $\overline{\alpha} \in \mathbb{C}$ is also a root of $p(x)$.

Proof. *Since $\alpha \in \mathbb{C}$ is a solution (root) of the equation $p(x) = 0$,*

$$p(\alpha) = a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0.$$

Let us take the conjugate of α, $\overline{\alpha}$. Let us see that it is also a root of the previous equation. First of all, notice that $\overline{\alpha}^n = \overline{\alpha^n}$, and that since the coefficients of $p(x)$ are real, we can conclude $\overline{a_k} = a_k$, for any k. Thus,

$$p(\overline{\alpha}) = a_n\overline{\alpha}^n + a_{n-1}\overline{\alpha}^{n-1} + \cdots + a_1\overline{\alpha} + a_0 =$$

$$a_n\overline{\alpha^n}+a_{n-1}\overline{\alpha^{n-1}}+\cdots+a_1\overline{\alpha}+a_0=\overline{a_n\alpha^n}+\overline{a_{n-1}\alpha^{n-1}}+\cdots+\overline{a_1\alpha}+\overline{a_0}=\overline{p(\alpha)}=0.$$

Consequently, in a real polynomial equation $p(x) = 0$*, the non-real roots appear in pairs. More precisely, the non-real roots appear as a number and its conjugate. Hence,*

$$p(x)=(x-\beta_1)\dots(x-\beta_k)(x-\alpha_1)(x-\overline{\alpha_1})\dots(x-\alpha_l)(x-\overline{\alpha_l}),\ \textit{with}\ \beta_1,\dots,\beta_k\in\mathbb{R}.$$

Finally notice that

$$(x-\alpha_i)(x-\overline{\alpha_i})=x^2-(\alpha_i+\overline{\alpha_i})x+\alpha_i\overline{\alpha_i}$$

is a quadratic polynomial with real coefficients.