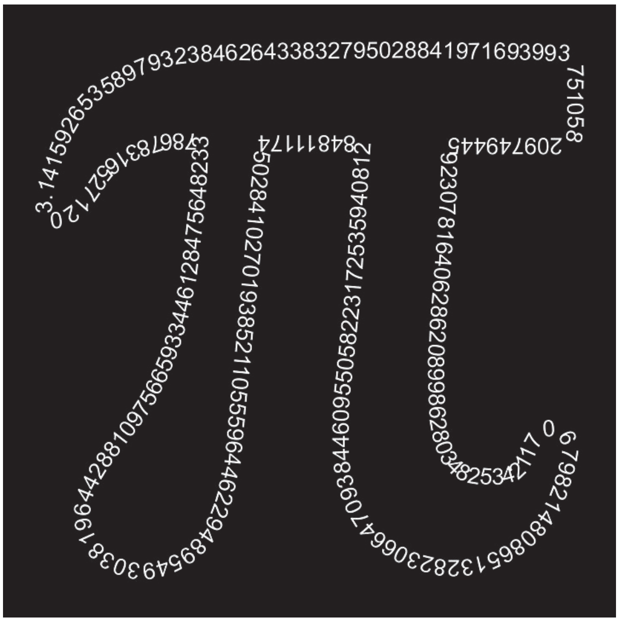
MATHS BASIC COURSE FOR UNDERGRADUATES



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1st SUBJECT: SET THEORY

1 Sets. Sets of numbers

Definition. A set *is a collection of objects. Those objects are called the elements of the set. Normally, the sets are represented by capital letters and the elements by small letters. In addition to this, to represent that the element* a *belongs to the set* A*, we write* $a \in A$*, and in the opposite case, we write* $a \notin A$.

The simplest sets are the numeral systems or the sets of numbers. Let us introduce the following ones:

$\mathbb{N}$: The set of the natural numbers. We use those numbers to count. $1, 2, 3, 4, \ldots$. Note that $0 \notin \mathbb{N}$.

$\mathbb{Z}$: The set of the integer numbers. It includes all the natural numbers, the zero number and the negative of all the natural numbers: $\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$

$\mathbb{Q}$: The set of the rational numbers, which is formed by all the allowed fractions of all the integer numbers: a/b. Remind that b must be different from 0 and that $a/b = c/d$ if and only if $ad = bc$.

It is obvious that any $a \in \mathbb{Z}$ integer number is also a rational number, since $a = \frac{a}{1} \in \mathbb{Q}$. Then $\mathbb{Z} \subseteq \mathbb{Q}$. Note that a rational number can be represented by different equivalent fractions, as well. According to this, if we multiply both terms, numerator and denominator of a fraction by the same non zero integer number the obtained fraction is equivalent to the initial one, and usually the irreducible form for which the numerator and denominator are coprime numbers is used to represent a fraction.

$\mathbb{I}$: The set of the irrational numbers, i.e. the set of the non rational numbers. It is the set formed by all the numbers whose representation in the decimal system has infinite non repetitive digits in its decimal part. In other words, the elements of such set can not be represented by the quotient of two integer numbers. It is obvious that $\mathbb{Q} \cap \mathbb{I} = \emptyset$. For instance, the following ones are irrational numbers: $\sqrt{3} = 1,7320508...$, $\pi = 3,14159265....$, which is the proportion between the length and the diameter of the circumference or the number $e = lim_{n \longrightarrow \infty}(1 + \frac{1}{n})^n$, i.e. the number $2,71828182845905...$

$\mathbb{R}$: The set of the real numbers. It not easy to prove how they are built from the rational numbers. Anyway, the use of the real numbers is very common, since they are used as measurement. The real numbers are addressed in an infinite line, called the real line. On the other hand, remind that any real number can be represented by its decimal representation, and that the union between the set of the rational numbers and the set of the irrational numbers completes the set of the real numbers.

$\mathbb{C}$: The set of the complex numbers. By definition, a complex number corresponds to a representation: $a+bi$, where $a, b \in \mathbb{R}$. More important than knowing their meaning is knowing how we can compute them, using additive notation or multiplicative notation. The addition of two complex numbers is described as follows:

$$(a+bi)+(c+di)=(a+c)+(b+d)i,$$

"component by component", and the product of any two complex numbers is described as follows:

$$(a+bi).(c+di)=(ac-bd)+(ad+bc)i.$$

It is not compulsory to memorize the previous formula. It is enough to keep in mind that $i^2=-1$ and to do the product in a common way.

Remark. (i) *The symbol $*$ over a set expresses that the element 0 is removed. For instance, $\mathbb{Z}^*=\mathbb{Z}-\{0\}$.*

(ii) *The symbol $+$ over a set expresses that the numbers involved in that set are greater than or equal to 0. For instance, $\mathbb{R}^+=\{a \in \mathbb{R} \mid a \geq 0\}$.*

Ways of defining sets. We distinguish the following ones:

(i) Enumerating all the elements of the set between two braces. For example, $A = \{1, 2, 3, 4\}$.

(ii) Giving a property (-ies) or a characteristic (-s) that the elements of the set satisfies. For instance, the previous set is described as follows: $A=\{x \in \mathbb{Z} \mid 1 \leq x \leq 4\}$.

(iii) When the elements of the set are obtained giving values to some parameters. Thus, in the case of the set formed by the even numbers, it can be represented by $\{2n \mid n \in \mathbb{Z}\}$, and the set $\{(\lambda, \mu, -\lambda-\mu) \mid \lambda, \mu \in \mathbb{R}\}$ represents the plane $x+y+z=0$ in $\mathbb{R}^3$.

To tell the truth, the third option is a particular case of the second one. For instance, the set $\{2n \mid n \in \mathbb{Z}\}$ can be represented as well by $\{x \in \mathbb{Z} \mid \exists \lambda \in \mathbb{Z}$ such that $x=2\lambda\}$, and in that case a property satisfied by the elements of the set is given.

Definition. *If a set does not have any element, the set is called "empty set" and it is represented by the symbol $\varnothing$.*

Definition. *Let A and B any two sets.*

(i) *B is said to be a subset of A (or it is said that B is contained in A), if all the elements of B are elements of A, as well. In order to express this content, $B \subseteq A$ is written.*

(ii) *B is a proper subset of A; it means that B is a subset of A but not equal to A. To express the proper content, $B \subsetneq A$ is written.*

(iii) *We say that the set A and the set B are equal if both sets have the same elements. In other words, $a \in A \iff a \in B$. Thus, $A = B \iff A \subseteq B$ and $B \subseteq A$, and consequently $A \neq B$ if and only if there exists some $a \in A$ such that $a \notin B$ or there exists some $b \in B$ such that $b \notin A$.*

Proposition. *Let A be a set. Then $\varnothing \subseteq A$.*

Example. *Let $A = \{1, 2, 3\}$ be a set. It is clear that $\{1, 2\} \subseteq A$ and that $\{1, 4\} \nsubseteq A$. We enumerate now all the subsets of A:*

$$\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \text{ and } \{1,2,3\}.$$

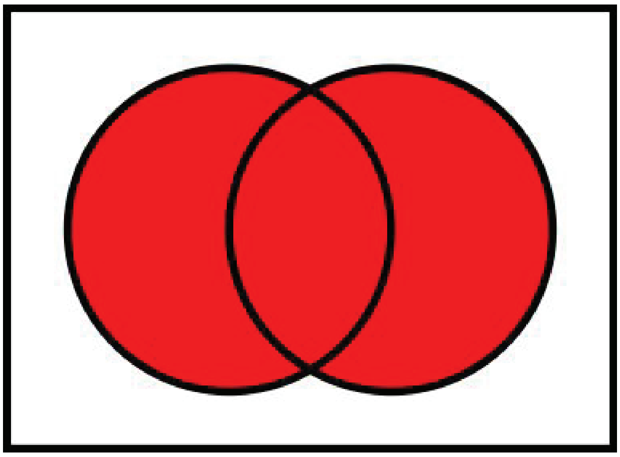
Thus the set A has 8 subsets.

Definition. *Let A be a set. The number of elements of A is called the cardinal of A, and it is denoted by the symbol $|A|$. If the cardinal of a set is not finite, it is said that the set is infinite.*
For instance, $|\mathbb{Z}| = \infty$, $|\varnothing| = 0$ and $|\{1, 2, 3, 4\}| = 4$.

Set operations. Let A and B be two sets. We define the following operations:

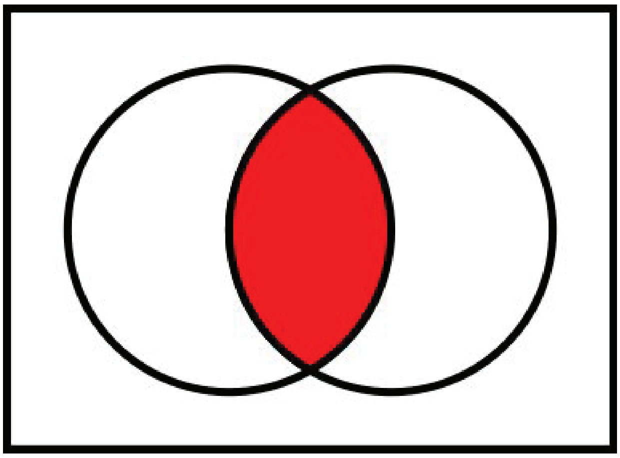
(i) The *union* between A and B, which is denoted by $A \cup B$. (It is read as "A union B"):

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$



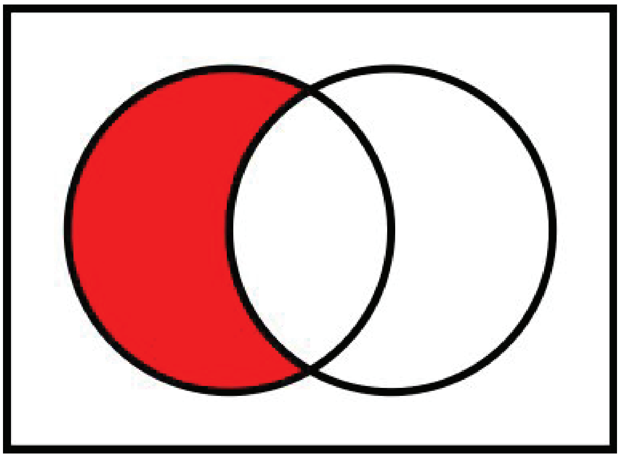
(ii) The *intersection* between A and B, which is denoted by $A \cap B$. (It is read as "A intersection B"):

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$



(iii) The *difference* between A and B, which is denoted by $A - B$. (It is read as "A minus B"):

$$A - B = \{x \mid x \in A \wedge x \notin B\}.$$



Those previous operations can be addressed graphically. They are called *Venn's diagrams* and they are pretty useful to announce about the properties of the set operations. Keep in mind that $A \cup B = B \cup A$ and that $A \cap B = B \cap A$, but however $A - B$ can be different from $B - A$. In addition to this, $|A \cup B| = |A| + |B| - |A \cap B|$. What's more, although some properties are obviously noticed from the diagrams, in mathematics the graphics are not in general enough to ensure the proofs. In other words, a formal proof is always necessary.

According to this, to prove the statements of the following theorem it would be recommendable to represent the corresponding Venn's diagrams to convince ourselves about the results, and later on to give the formal proofs based on the definitions. The proof of this theorem, as some other proofs of theorems included in this chapter are left to the reader.

Theorem. *(Distributive property). Let A, B and C be sets. Then the following properties hold:*

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. *(The union is distributive respect to the intersection.)*

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. *(The intersection is distributive respect to the union.)*

Universal set. In general, the used sets are contained in a reference set, which is called the *universal set*. For instance, if we only work with the real numbers, then the universal set would the set of all the real numbers, i.e. $\mathbb{R}$.

Definition. *Let us assume that the universal set we are working with is the set X, and let A be a subset of X. The complement of A, which is denoted by A^c, corresponds to the following set:*

$$A^c = X - A = \{x \in X \mid x \notin A\}.$$

For instance, if $\mathbb{R}$ is the universal set, then $(0,1)^c = (-\infty, 0] \cup [1, +\infty)$ and $(-\infty, 0]^c = (0, +\infty)$.
It is obvious that $A \cap A^c = \emptyset$ and $A \cup A^c = X$. In the following theorem, the most important properties of a complement set are given.

Theorem. *(De Morgan's laws). Let A and B be two sets contained in the universal set. Then,*

(i) $(A \cup B)^c = A^c \cap B^c$.

(ii) $(A \cap B)^c = A^c \cup B^c$.

Before finishing this section, an important structure called the cartesian product is introduced. Remind that an ordered pair *is an expression of the type (a, b), and that two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$. In general, an expression of the type $(a_1, a_2, \ldots, a_n)$ is called an* ordered tuple *(or if we are interesed in emphasising the value of n, it is also called an (ordered) n-*tuple*).*
*If $1 \leq i \leq n$, the a_i element is called the i*th component *of the tuple.*

Definition. *Let A and B be two sets. The cartesian product of A and B, which is denoted by $A \times B$ is the set formed by all the possible ordered pairs, for which the first* *component belongs to the set A and the second component belongs to the set B. In other words,*

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Usually if $A_1, A_2, \ldots, A_n$ are n sets, the cartesian product of all those n sets corresponds to:

$$A_1 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, \forall i \in \{1, \ldots, n\}\}.$$

In the particular case that all the n sets are equal to A, the simpler expression A^n can be used instead of $A \times \overset{n}{\cdots} \times A$.

Note that the plane $\mathbb{R}^2$ and the space $\mathbb{R}^3$ are cartesian products. The name "cartesian" comes from the French mathematician named Descartes, since he was the first one to express the points of the plane and the points of the space by the expressions (a, b) and (a, b, c) respectively. This is also the reason why the descriptions of those points are called cartesian components*, and analogously the same happens to the name "cartesian product".*

Example. *If $A = \{1, 2, 3\}$ and $B = \{x, y\}$ then,*

$$A \times B = \{(1, x), (2, x), (3, x), (1, y), (2, y), (3, y)\}.$$

The cardinal of the cartesian product. *If A and B are two finite sets, then it is clear that $|A \times B| = |A||B|$, and in general, $|A_1 \times \cdots \times A_n| = |A_1| \cdots |A_n|$.*

2 Equivalence and order relations

Let A and B be two sets. As it has been defined before, the cartesin product of A and B corresponds to the set $\{(a, b) \mid a \in A, b \in B\}$, which is denoted by $A \times B$. By definition, a binary relation of (from) the set A on the set B is a subset (denoted by R) of the cartesian product $A \times B$.
To be more precise, a binary relation from the set A on the set B, which is denoted by $\Re$, is defined as follows:

$$a \Re b \Longleftrightarrow (a, b) \in \textit{ R}.$$

According to this, an element $a \in A$ is related (respect to the binary relation $\Re$) to an element $b \in B$ if and only if the ordered pair $(a, b) \in$ R. In such case, $a \Re b$ is written.

Remark. *A binary relation can be defined by a P property, as well.*

$$a \Re b \Longleftrightarrow P(a, b) \textit{ is true}$$

As a consequence, the subset R of $A \times B$ can be expressed as follows:

$$\textit{R} = \{(a, b) \in A \times B \mid P(a, b) \textit{ is true } \}.$$

Definition. *The domain and the rank of a binary relation $\Re$ are defined as follows:*

(i) *Domain*$(\Re) = \{a \in A \mid a\Re b,$ *for some* $b \in B\}$

(ii) *Rank*$(\Re) = \{b \in B \mid a\Re b,$ *for some* $a \in A\}$.

Definition. *The inverse of a binary relation $\Re$, which is denoted by $\Re^{-1}$, corresponds to the subset $\{(b, a) \in B \times A \mid (a, b) \in R\}$.*

Definition. *Let $\Re$ be a binary relation of A on B and S be a binary relation of B on C. The composition of the binary relations $\Re$ and S is a subset of $A \times C$ formed by the ordered pairs $(a, c) \in A \times C$, for which there exists an element $b \in B$ such that $a\Re b$ and bSc. Thus, it is said that $S \circ \Re$ is a binary relation of A on C.*

From now on, we will work with binary relations defined on the same set A, i.e. with subsets $R \subseteq A \times A$.

Definition. *Let A be a set. A binary relation $\Re$ of A on A is said to be a binary relation on A. The binary relation on A could be:*

(i) *Reflexive:* $\forall a \in A, a\Re a$.

(ii) *Symmetric:* $\forall a, b \in A$ *such that* $a\Re b$, *it holds also that* $b\Re a$.

(iii) *Transitive:* $\forall a, b, c \in A$ *such that* $a\Re b$ *and* $b\Re c$, *it holds also that* $a\Re c$.

(iv) *Antisymmetric:* $\forall a, b \in A$ *such that* $a\Re b$ *and* $b\Re a$, *it follows that* $a = b$.

(v) *Entire: for any* $a, b \in A$ *such that* $a \neq b$, *it holds* $a\Re b$ *or* $b\Re a$.

Definition. *Let $\Re$ be a binary relation on A.*

(i) *if $\Re$ is reflexive, antisymmetric and transitive, the relation $\Re$ is said to be an order relation on A. In this case, $(A, \Re)$ is said to be an ordered set.*

(ii) *if $\Re$ is reflexive, symmetric and transitive, the relation $\Re$ is said to be an equivalence relation on A.*

Example. *The following ones are ordered sets:*

$$(\mathbb{N}, \leq), (\mathbb{Z}, \leq), (\mathbb{Q}, \leq), (\mathbb{R}, \leq),$$

for which the symbol $\leq$ indicates the usual inequality "less than or equal to".

Examples. *The following binary relations are equivalence relations:*

(i) *in the set* $\mathbb{Z} \times \mathbb{Z}^*$, *let us define*

$$(a, b)\Re(c, d) \iff ad = cb.$$

(ii) *in the set* $\mathbb{Z}$, *let us define the congruence module* 2*: for any* $a, b \in \mathbb{Z}$,

$$a\Re b \iff b - a \text{ is a multiple of the integer number } 2.$$

(iii) *in the set* $\mathbb{Z}$, *for any* $n \in \mathbb{N}$ *let us define the congruence module* n, *as follows: for any* $a, b \in \mathbb{Z}$,

$$a\Re b \iff b - a \text{ is a multiple of the natural number } n.$$

Using the previous relation $\Re$, *if* $a\Re b$ *we usually express it by* $a \equiv b \pmod{n}$, *and it is read "*a *is congruent to* b *module* n*".*

Remark. *It holds*

$$a \equiv b \pmod{n} \iff \exists k \in \mathbb{Z} \text{ such that } b = a + nk.$$

Definition. *Let* $\Re$ *be an equivalence relation on the set* A. *For any* $a \in A$, *we define the equivalence class of* a, *which is denoted by* $\overline{a}$, *as the set formed by all the elements of* A *which are related with the element* a *by the relation* $\Re$. *In other words,*

$$\overline{a} = \{x \in A \mid x\Re a\}.$$

Any $x \in \overline{a}$ *is called a representative for the equivalence class* $\overline{a}$.

Proposition. *Let* $\Re$ *be an equivalence relation on the set* A. *For any* $a, b \in A$ *the following properties hold:*

(i) $a \in \overline{a}$ *(and consequently* $\overline{a} \neq \emptyset$*). In particular, any element of* A *belongs to some equivalence class.*

(ii) $\overline{a} = \overline{b} \iff a\Re b$.

(iii) $\overline{a} \cap \overline{b} = \emptyset \iff a$ *and* b *are not related by* $\Re$.

Definition. *Let* $\Re$ *be an equivalence relation on the set* A. *We define the quotient set of* A *by the relation* $\Re$, *which is denoted by* $A/\Re$, *as the set formed by all the possible equivalence classes of all the elements of* A;

$$A/\Re = \{\overline{a} \mid a \in A\}.$$

Example. *In the set* $\mathbb{Z} \times \mathbb{Z}^*$, *let us consider the following equivalence relation:*

$$(a, b)\Re(c, d) \iff ad = cb.$$

Then, $\overline{(a, b)} = \{(c, d) \in \mathbb{Z} \times \mathbb{Z}^* \mid \frac{a}{b} = \frac{c}{d}\}$, *and the quotient set related to this relation corresponds to* $(\mathbb{Z} \times \mathbb{Z}^*)/\Re = \mathbb{Q}$.