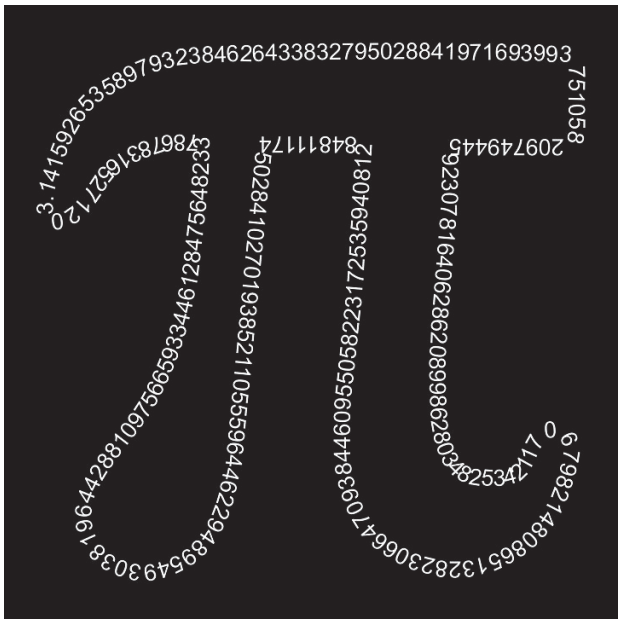
MATHS BASIC COURSE FOR UNDERGRADUATES



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PRELIMINARY SECTION

MATHEMATICAL LANGUAGE

1 First concepts

Axiom: An axiom is a statement that is taken to be true, to serve as a premise or starting point for further reasoning and arguments.

Definition: The act of defining, or of making something definite, distinct, or clear. A definition is not a specific reference to a mathematical object, but more truly it is a reference to the language of mathematics that we use to define mathematical objects.

Theorem: In mathematics, a theorem is defined as a statement that has been proven on the basis of previously established statements, such as other theorems, and generally accepted statements, such as axioms. A theorem is a logical consequence of those axioms. The proof of a mathematical theorem is a logical argument for the theorem statement given according to the rules of a deductive system.

Proposition: In mathematics, a proposition is a formal statement of either a truth to be proved or an operation to be perfomed; a theorem or a problem.

Corollary: In mathematics, a corollary usually follows a theorem. The use of the term "Corollary" rather than "Proposition" or "Theorem" is subjective. The importance of the corollary is often considered secondary to that of the initial theorem.

Symbol: A symbol is a mark or a sign that indicates or represents an idea, object, or a relationship. For example, numerals are symbols for numbers. Alphabetic letters may be symbols for sounds. The variable "x", in a mathematical equation, may signify for instance the position of a particle in the space, etc.

2 Some useful symbols in Mathematics

- **Set theory.** Let x be an element and let A, B any two subsets.

Operation	Notation	How do we read it?
Content	$x \in A$	x is contained in A or x belongs to A
Content	$A \subseteq B$	A is a subset of (the set) B or A is contained in (the set) B
Content	$A \subset B$	A is a proper subset of B or A is contained properly in B

A twisted line over a symbol undoes its meaning. For instance, $x \notin A$ means that x does not belong to A.

- **Expressions**

Operation	Notation	How do we read it?
Equal	$x = y$	x is equal to y
Less than	$x < y$	x is strictly less than y
Greater than	$x > y$	x is greater than y
Equivalent	$x \approx y$	x is equivalent to y

- **Basic operators**

Let p and q any two propositions.

Operation	Notation	How do we read it?
Negation	$\neg p$	Not p
Meet	$p \wedge q$	p and q
Join	$p \vee q$	p or q

- **Consequence**

What does "$p \Longrightarrow q$" mean? If the proposition "p" holds, then the proposition "q" is satisfied as well. What is more, if both "$p \Longrightarrow q$" and "$q \Longrightarrow p$" hold, then it is read "p" holds if and only if "q" holds.

- **Quantifiers**

The quantifiers are necessary to confirm when a statement is true or not. There are three basic quantifiers: the universal quantifier, the existential quantifier, and the existential quantifier with the uniqueness mark.

Operation	Notation	How do we read it?
Universal	$\forall x..$	for all x
Existential	$\exists x..$	there exists at least x for which
Existential with uniqueness mark	$\exists! x$	there exists only one x for which

3 Mathematical proofs

Let us assume that the statement p is true. Now the aim is to prove that a statement called q holds, as well. In other words, $p \Longrightarrow q$. The process consists in obtaining a chain of conclusions, starting from the statement p as follows:

$$p \Longrightarrow p_1, p_1 \Longrightarrow p_2, \ldots, p_n \Longrightarrow q,$$

for which each statement p_i is a previously given hypothesis or an already proven theorem. When that chain reaches $p_n \Longrightarrow q$, the statement q is obtained, and the process finishes. Next, we enumerate the most used mathematical proof methods.

(i) Direct proof

(ii) Indirect proof

(iii) Finding a counter example

(iv) Proof by recursion or by induction

(i) Direct proof: if the statement p is true and the consequence $p \Longrightarrow q$ holds, then the statement q is true, as well.

(ii) Indirect proof. There are two types of indirect proofs:

ii-a) By contrast: to prove the result "if p then q", the contrary is proved: "$\neg q$ then $\neg p$".

ii-b) By way of contradiction: verifying the veracity of q consistes in imposing its negation and getting as a consequence a contradiction with the initial hypotheses; it means that $\neg q$ is also false, and consequently it is deduced that q is true.

Example. *If S is the set of all prime numbers, then S is an infinite set. ($p \Longrightarrow q$).*

By way of contradiction, let us suppose that S is a finite set, i.e. $p \wedge \neg q$. Let us consider $S = \{p_1, p_2, \ldots, p_k\}$. Since the set S is finite, we compute the product of all the elements of S, and denote by b the following number: $b = (p_1.p_2 \ldots p_k) + 1$. It is clear that there exists at least a p' prime number such that p' divides b (or such that b is a multiple of that p' prime number). Let us denote by r that statement. Since p' is a prime number and S the set of all prime numbers, it is obvious that $p' \in S$. On the other hand, looking at the definition of b, we conclude that neither of the elements of S divides b (or that b is not multiple of neither of the elements of S). In conclusion, p' does not divide b, i.e. $\neg r$ holds, and we get a contradiction, i.e. $r \wedge \neg r$. In other words,

$(p \wedge \neg q) \Longrightarrow (r \wedge \neg r)$, which is false. This implies that the set S of all prime numbers is not finite, i.e. S is an infinite set.

(iii) Finding a counter example: in this case, to prove the negation of the conclusion "$p \Longrightarrow q$", an example for which, at the same time p and $\neg q$ hold must be found.

Example. *Let us consider the following p and q propositions. p proposition: the integer number n is divisible by 6 and by 4, and q proposition: the integer number n is divisible by 24. Does p proposition imply q proposition?*

The answer is no. For instance, the integer number 12 satisfies at the same time the statements p and $\neg q$, since 12 is divisible by 6 and by 4, but it is not divisible by 24. In conclusion, p does not imply q.

(iv) Proof by recurrence or by induction. These arguments prove that the proposition for which any natural number n is involved is true. To prove it, it is enough to check that such proposition holds for the natural number 1, and that if the statement of the proposition holds true for the natural number n, then it also holds for the following natural number $n + 1$.
Symbolically, the induction proposition is the following one:

$$p(1) \wedge \forall k[p(k) \Longrightarrow p(k+1)] \Longrightarrow \forall n, p(n).$$

If it is possible to prove the previous statement, then also $p(1) \wedge \forall k[p(k) \Longrightarrow p(k+1)]$ holds, and consequently the statement $\forall n, p(n)$ is true, as well.

The induction method consists of two steps:

(i) **First step**. Prove that the statement $p(1)$ is true.

(ii) **Induction step**. Prove that $\forall k[p(k) \Longrightarrow p(k+1)]$.

Example. *Prove the following statement:*

$$\forall n, 2^n \leq 2^{n+1}$$

(i) **First step**. *Prove that the statement $p(1)$ holds: $2^1 \leq 2^{1+1}$, since $2^1 = 2$, $2^{1+1} = 4$ and $2 \leq 4$.*

(ii) **Induction step**. *Prove that $\forall k[p(k) \Longrightarrow p(k+1)]$. Let us assume that the statement $p(k)$ is true, i.e. let us assume that $2^k \leq 2^{k+1}$ holds (hypothesis). Now let us prove that the statement $p(k+1)$ is true, i.e. let us prove that $2^{k+1} \leq 2^{k+1+1} = 2^{k+2}$ holds. To do this, let us multiply both sides of the inequality of the hypothesis by 2. Then the statement $2^k.2 \leq 2^{k+1}.2$ is satisfied, in other words, $2^{k+1} \leq 2^{k+2}$ holds, which is what we desired.*