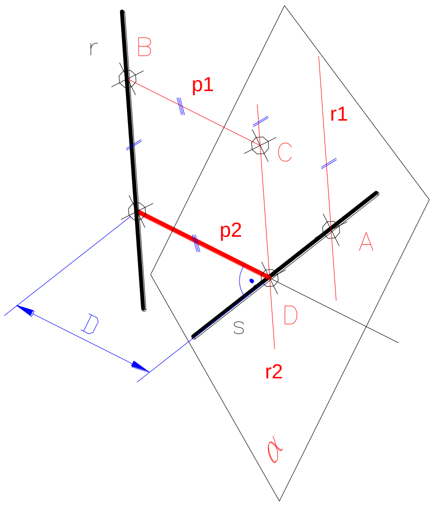

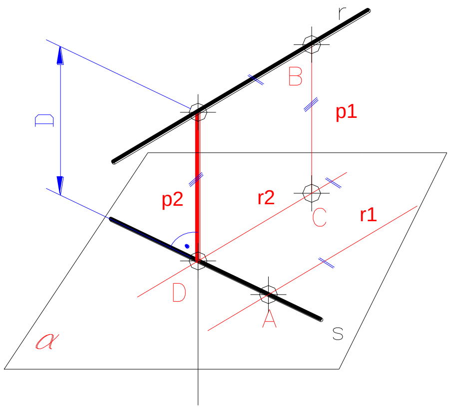

4.8.A – Distance between two skew lines

The distance between two skew lines r and s is the distance between the plane that passing through the line r is parallel to the line s and the plane that passing through the line s is parallel to the line r.

Vector expression

Let *r* and *s* be two lines defined by $r(A_r,\vec{u}_r)$ and $s(A_s,\vec{u}_s)$. The distance between the lines *r* and *s*, is the distance from the point A_s to the plane $\alpha(A_r,\vec{u}_r,\vec{u}_s)$: $d(r,s)=d(A_s,\alpha)$

The vector expression of the distance from the point *P* to the plane α is given by:

$$d(P,\alpha)=\frac{|\overrightarrow{A_\alpha P}\cdot\vec{n}_\alpha|}{|\vec{n}_\alpha|}$$

In our case, by considering $A_\alpha = A_r$, $P=A_s$ and $\vec{n}_\alpha=\vec{u}_r\times\overrightarrow{u_s}$ we get the following expression: $d(r,s)=d(A_s,\alpha)=)=\dfrac{|\overrightarrow{A_rA_s}\cdot(\vec{u}_r\times\vec{u}_s)|}{|\vec{u}_r\times\vec{u}_s|}$

Using the expression of the mixed product we obtain:

$$\boxed{d(r,s)=\frac{|det(\overrightarrow{A_rA_s},\vec{u}_r,\vec{u}_s)|}{|\vec{u}_r\times\vec{u}_s|}} \quad (3)$$

Analytic expression:

Let assume that the continuous equations of the lines are:

$$r:\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \qquad s:\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$

The vectors that will be used in expression (3) are:

$\vec{u}_r = (a_1, b_1, c_1),\ \vec{u}_s = (a_2, b_2, c_2),\ \overrightarrow{A_r A_s} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

As the formula obtained using these values is complicated, in practice expression (3) is applied using the numerical values of the problem.

If the lines are not given using their continuous form, it is necessary to obtain a point and the direction vector of each of the lines. The expression (3) is applied afterwards.

Remark: In some cases it is easier to develop the definition of the distance between two lines. In that case it is enough to consider the point A_s and to calculate the plane α that passing through the point $A_r(x_1, y_1, z_1)$ contains the line *r* and it is parallel to the line *s*. Next, we calculate the distance from the point A_s to the plane α. The equation of the cited plane α can be calculated using the determinant:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

►***Example 34 (A)***

Calculate the distance between the lines $r: \dfrac{x-13}{4} = \dfrac{y}{5} = \dfrac{z-5}{-3}$ and $s: \begin{cases} x = 3 + 5t \\ y = 2 + t \\ z = 4t \end{cases}$.

Solution:

In this example, we will apply the analytic formula to calculate the distance between two skew lines.

The line *r* is determined by the point $B(13, 0, 5)$ and the direction vector $\vec{u}_r = (4, 5, -3)$. And the line *s* is determined by the point $A(3, 2, 0)$ and the direction vector $\vec{u}_s = (5, 1, 4)$ The distance between both lines is:

$$d(r,s) = \frac{|\,det(\overrightarrow{AB}, \vec{u}_r, \vec{u}_s)\,|}{|\,\vec{u}_r \times \vec{u}_s\,|} = \frac{187}{\sqrt{1931}}$$

being:

$\overrightarrow{AB} = B - A = (10, -2, 5)$,

$$det(\overrightarrow{AB}, \vec{u}_r, \vec{u}_s) = \begin{vmatrix} 10 & -2 & 5 \\ 4 & 5 & -3 \\ 5 & 1 & 4 \end{vmatrix} = 187,$$

$|\,\vec{u}_r \times \vec{u}_s\,| = |\,(23, -31, -21)\,| = \sqrt{1931}$

Mutual perpendicular of two lines

The mutual perpendicular of two skew lines is the line that intersects perpendicularly each of the given lines. The mutual perpendicular p of two lines r and s is determined by the intersection of the panes $\alpha(A_r,\vec{u}_r,\vec{u}_r\times\vec{u}_s)$ and $\beta(A_s,\vec{u}_s,\vec{u}_r\times\vec{u}_s)$.

As a result, the analytic expression of the mutual perpendicular is:

$$p:\begin{cases} det(\overrightarrow{A_rX},\vec{u}_r,\vec{u}_r\times\vec{u}_s) &= 0 \\ det(\overrightarrow{A_sX},\vec{u}_s,\vec{u}_r\times\vec{u}_s) &= 0\end{cases}$$

, being X any point of the mutual perpendicular p.

The distance between two skew lines is the distance between the points of intersection of the mutual perpendicular with each of the given lines.

Another way to obtain the mutual perpendicular is by using the technique of generic points:

Let P_r and P_s be the intersection points of the mutual perpendicular with the skew lines r and s. The coordinates of the point P_r will be the parametric equations of the line r, that is to say: $P_r(x_1+a_1t, y_1+b_1t, z_1+c_1t)$. Similarly, the coordinates of the point P_s will be the parametric coordinates of the line s: $P_s(x_2+a_2s, y_2+b_2s, z_2+c_2s)$

As the vector $\overrightarrow{P_rP_s}$ is perpendicular to $\vec{u}_r$ and $\vec{u}_s$, their scalar products are zero:

$$\begin{cases}\overrightarrow{P_rP_s}\cdot\vec{u}_r &= 0\\ \overrightarrow{P_rP_s}\cdot\vec{u}_s &= 0\end{cases}$$

By doing the calculations, a system of two equations and two unknowns (*t* and *s*) is obtained.

After solving the system, the values of the parameters *t* and *s* are substituted in P_r and P_s, obtaining the coordinates of these two points.

Once the coordinates of the points P_r and P_s are known, the equation of the mutual perpendicular p can be calculated. The direction vector of the mutual perpendicular is $\overrightarrow{P_rP_s}$. The distance between the lines *r* and *s* is given by the modulus $|\overrightarrow{P_rP_s}|$.

►***Example 35 (A)***

Obtain the equation of the mutual perpendicular of the lines $r: x = y = z$ and $s: x = y = 3z - 1$.

Solution: Using the equations of the given lines we conclude:
$A_r(0,0,0), \vec{u}_r = (1,1,1), A_s(-1,-1,0), \vec{u}_s = (3,3,1) \to \vec{u}_r\times\vec{u}_s = (-2,2,0)$

The mutual perpendicular p is determined by the intersection of the planes:

$$det(\overrightarrow{A_rX},\vec{u}_r,\vec{u}_r\times\vec{u}_s)=0\rightarrow\begin{vmatrix}x & y & z\\ 1 & 1 & 1\\ -2 & 2 & 0\end{vmatrix}=0\leftrightarrow x+y-2z=0$$

$$det(\overrightarrow{A_sX},\vec{u}_s,\vec{u}_r\times\vec{u}_s)=0\rightarrow\begin{vmatrix}x+1 & y+1 & z\\ 3 & 3 & 1\\ -2 & 2 & 0\end{vmatrix}=0\leftrightarrow x+y-6z+2=0$$

As a consequence, the equation of the mutual perpendicular is: $p:\begin{cases}x+y-2z & =0\\ x+y-6z+2 & =0\end{cases}$.

Using the technique of generic points of each of the lines, we have:

$P_r=(t,t,t),\ \ P_s=(3s-1,3s-1,s)\quad \overrightarrow{P_rP_s}=(3s-t-1,3s-t-1,s-t)$

This vector must be perpendicular to the direction vectors $\vec{u}_r$ and $\vec{u}_s$:

$\overrightarrow{P_rP_s}\cdot\vec{u}_r=(3s-t-1,3s-t-1,s-t)\cdot(1,1,1)=0$

$\overrightarrow{P_rP_s}\cdot\vec{u}_s=(3s-t-1,3s-t-1,s-t)\cdot(3,3,1)=0$

Doing some calculations we get: $\begin{cases}7s-3t & =2\\ 19s-7t & =6\end{cases}\leftrightarrow\begin{cases}t & =\dfrac{1}{2}\\ s & =\dfrac{1}{2}\end{cases}$

The points that correspond to the values of the parameters *t* and *s* are $P_r=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ and $P_s=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$.

Both points are the same, which means that the lines intersect each other, being the distance between them zero.

The mutual perpendicular passes through the point $P_r=(1/2,1/2,1/2)$ and its direction vector is $\vec{u}_r\times\vec{u}_s=(-2,2,0)$. So the equation of the mutual perpendicular is:

$$p:\frac{x-\frac{1}{2}}{-2}=\frac{y-\frac{1}{2}}{2}=\frac{z-\frac{1}{2}}{0}\quad\Rightarrow\quad p:\begin{cases}x+y-1=0\\ z=\dfrac{1}{2}\end{cases}$$