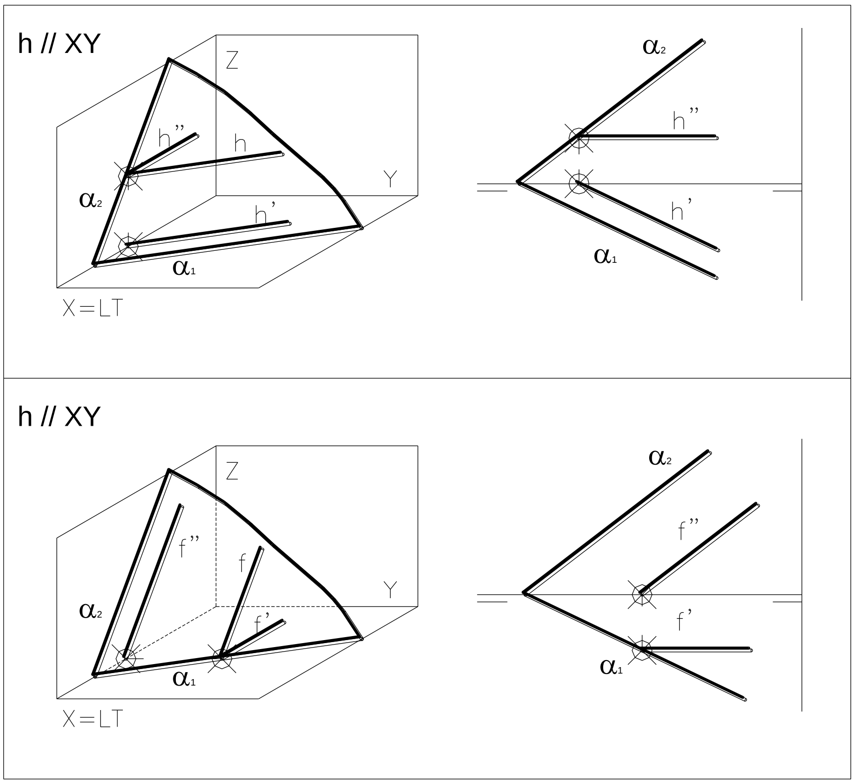
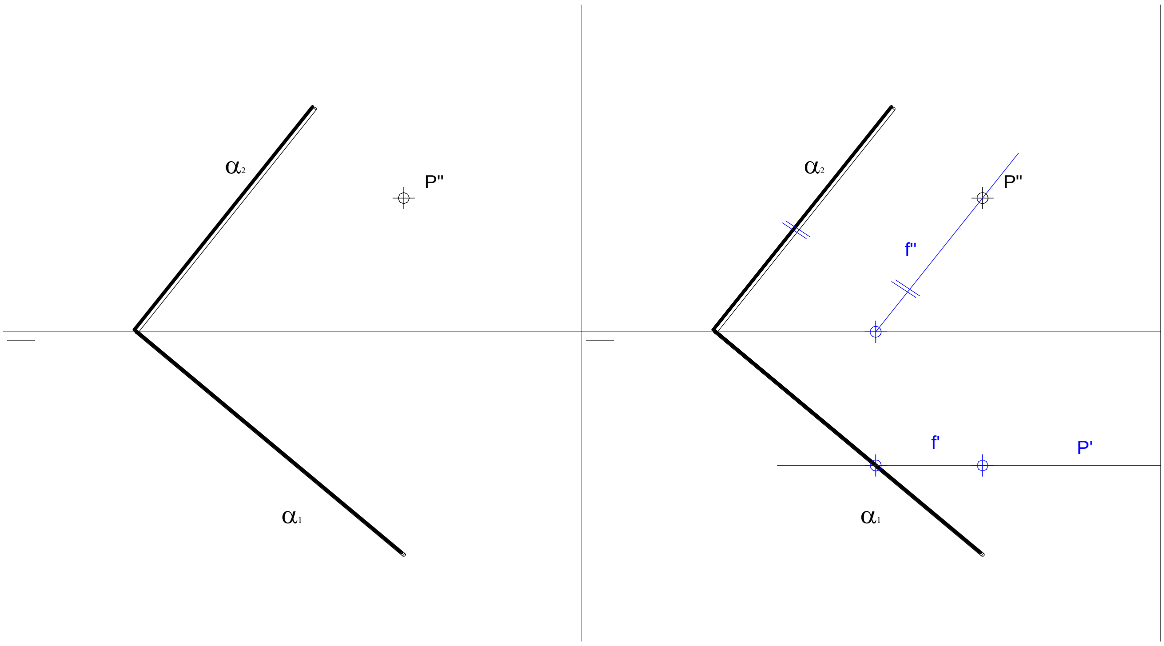
Among the infinite lines included in a plane, some of them are very important to define a plane in the diedric system. For example, the frontal lines (parallel to the PV), and the horizontal lines (parallel to the PH). One of the two projections of these two lines will be parallel to the trace of the plane in which it is located (in the horizontal line h, h' will be // to α_1 ; in the frontal line f, f'' will be // to α_2 .



►*Example 2 (G)*

Draw from the point P a frontal line of the plane. Find the projection of the point that is missing.



1.5.A - Plane

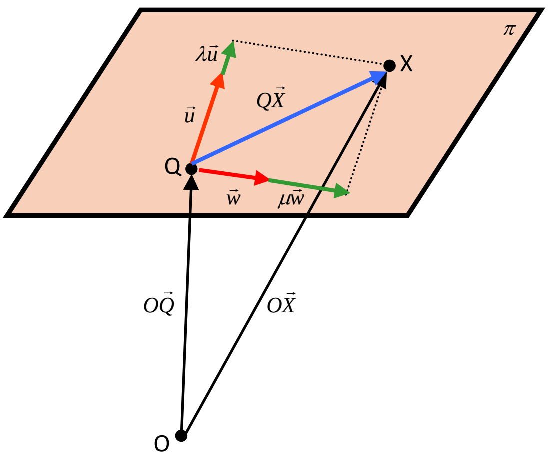
A point and two directions determine a plane. These directions are defined by any two linearly independent vectors included in the plane. These vectors are called direction vectors of the plane.

In the reference system $R = \{O\ ;\ \vec{x}, \vec{y}, \vec{z}\}$, the equations of the plane π that passing through the point $Q = (q_1, q_2, q_3)$, has as direction vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{w} = (w_1, w_2, w_3)$ are the following:

- <u>Vector equation of a plane</u>

As it can be seen in the figure, given any point X of the plane π, the vector $\overrightarrow{QX}$ can be written as linear combination of the vectors $\vec{u}$ and $\vec{w}$. Therefore, $\overrightarrow{QX} = \lambda\vec{u} + \mu\vec{w}$ being $\lambda, \mu \in \mathbb{R}$. In addition, $\overrightarrow{OX} = \overrightarrow{OQ} + \overrightarrow{QX}$.

By considering $\overrightarrow{OX} = \vec{x}$ and $\overrightarrow{OQ} = \vec{q}$, the expression $\vec{x} = \vec{q} + \lambda\vec{u} + \mu\vec{w}$ is obtained. And this is the vector equation of the plane.



- <u>Parametric equations of a plane</u>

The parametric equations of a plane are obtained by substituting the coordinates of the vectors in the vector equation of the plane:

$$\pi : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} + \lambda \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \mu \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

- <u>Implicit equation of a plane</u>

By considering the parametric equations as a linear system with two unknowns:

$$\begin{cases} x-q_1 = u_1\lambda + w_1\mu \\ y-q_2 = u_2\lambda + w_2\mu \\ z-q_3 = u_3\lambda + w_3\mu \end{cases}$$

The system will be a determinate compatible system if the following is satisfied:

$$\begin{vmatrix} x-q_1 & y-q_2 & z-q_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} x-q_1 & u_1 & w_1 \\ y-q_2 & u_2 & w_2 \\ z-q_3 & u_3 & w_3 \end{vmatrix} = 0$$

By developing any of the previous determinants, the implicit equation of the plane π is obtained:

$$\pi: Ax + By + Cz + D = 0$$

- Equation of a plane that contains a point

If the associated vector or normal vector of the plane (perpendicular vector to the plane), $\vec{n} = (A, B, C)$, and a point $P = (x_0, y_0, z_0)$ in the plane are known, the equation of the plane is given by:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

As a particular case, the equations of some planes which are parallel to the coordinate planes or to the coordinate axes are provided:

Equations of the planes that are parallel to the coordinate planes

Plane that is parallel to the plane XOY: $\Rightarrow z = constant$

Plane that is parallel to the plane XOZ: $\Rightarrow y = constant$

Plane that is parallel to the plane YOZ: $\Rightarrow x = constant$

Equations of the planes that are parallel to the coordinate axes

Plane that is parallel to the OX axis: $By + Cz + D = 0$

Plane that is parallel to the OY axis: $Ax + Cz + D = 0$

Plane that is parallel to the OZ axis: $Ax + By + D = 0$

►***Example 3 (A)***

Obtain the equations of the plane that contains the points $A = (5,4,2)$, $B = (1,6,4)$ and $C = (4,2,5)$.

Solution: The direction vectors of the plane are the following: $\vec{u} = B - A = (-4,2,2)$ and $\vec{w} = B - C = (3,-4,1)$.

- Vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$$

- Parametric equations

$$\begin{cases} x = 5 - 4\lambda + 3\mu \\ y = 4 + 2\lambda - 4\mu \\ z = 2 + 2\lambda + \mu \end{cases}$$

- Implicit equation

The implicit equation is obtained by solving the following equality: $\begin{vmatrix} x-5 & y-4 & z-2 \\ -4 & 2 & 2 \\ 3 & -4 & 1 \end{vmatrix} = 0$

Therefore, the implicit equation of the plane is $x + y + z = 11$.

- Normal equation

The normal vector of the plane is obtained doing the vector product of the direction vectors:

$$\vec{n} = \vec{v} \wedge \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 2 & 2 \\ 3 & -4 & 1 \end{vmatrix} = 10\vec{i} + 10\vec{j} + 10\vec{k} \Rightarrow \vec{n} = (10,10,10)$$

Or we can take $\vec{n} = (1,1,1)$ as the normal vector of the plane, because both vectors have the same direction.

As a point of the plane is known, for example by considering $A = (5,4,2)$, the normal equation of the plane is obtained:

$$1 \cdot (x-5) + 1 \cdot (y-4) + 1 \cdot (z-2) \Rightarrow x + y + z - 11 = 0$$