# 3. Gaia: Programen Egiaztapena

## 5. Ariketa-orria: Programa errekurtsiboak

1. Formulatu indukzio-hipotesi bat Hurrengo dei errekurtsiboentzat.

1.1. $F$ funtzioaren goiburukoa eta espezifikazioa:

```
function F(x: Integer) return z: Integer;
```

Aurre $\equiv \{ \ x > 0 \ \}$
Post $\equiv \{ \ z = \lfloor log_2 \ x \rfloor \ \}$ - - $x$-ren 2 oinarriko logaritmoaren
- - zati osoa da $z$

Kasu induktiboan egiten den deia:

```
w := F(x/2);
```

*Soluzioa*:

**(I.H.)** $\{ \ \frac{x}{2} > 0 \ \}$ `w := F(x/2);` $\{ \ w = log_2 \ \frac{x}{2} \ \}$

1.2. *konb* funtzioak $\binom{m}{n}$ zenbaki konbinatorioa kalkulatzen du (postbaldintzan $\binom{m}{n}$ zenbakia faktorialak erabiliz adierazten da).

```
function konb(m,n: Integer) return z: Integer is
```

**Aurre** $\equiv \{ \ 1 \leq n \leq m \ \}$

```
  if n = 1 then
     z := m;
  else
     zlag := konb(m,n-1);
     z := (zlag/n)*(m-n+1);
  end if;
```

**Aurre** $\equiv \{ \ z = \frac{m!}{n! \times (m-n)!} \ \}$

*Soluzioa*:

**(I.H.)** $\{ \ 1 \leq n-1 \leq m \ \}$ `zlag := konb(m,n-1);` $\{ \ zlag = \frac{m!}{(n-1)! \times (m-(n-1))!} \ \}$

2. Hurrengo programetan postbaldintza egoki bat asmatu.

2.1. *biderka* funtzioak bi zenbaki arrunten biderkadura kalkulatzen du.

```
function biderka(x,y:  Integer) return z:  Integer is
```
**Aurre** $\equiv \{\ x \geq 0 \wedge y \geq 0\ \}$
```
   if x = 0 or y = 0 then
      z := 0;
   elsif x = 1 then
      z := y;
   else
      z := biderka(x/2,2*y);
      if x mod 2 /= 0 then
         z := z+y;
      end if;
   end if;
```
**Post** $\equiv$

*Soluzioa*:

**Post** $\equiv \{\ z = x \times y\ \}$

2.2. *div* funtzioak bi zenbaki arrunten zatidura eta hondarra kalkulatzen ditu.

```
function div(x,y:  Integer) return z,h:  Integer is
```
**Aurre** $\equiv \{\ x \geq 0 \wedge y > 0\ \}$
```
   if x < y then
      z := 0;
      h := x;
   else
      (z,h) := div(x-y,y);
      z := z+1;
   end if;
```
**Post** $\equiv$

*Soluzioa*:

**Post** $\equiv \{\ z = \frac{x}{y}\ \wedge\ h = x\ mod\ y\ \}$

3. Hurrengo programa errekurtsiboetan kasu induktiboen frogapena egin.

3.1. *dig_hand* funtzioak zenbaki arrunt baten digitu handiena kalkulatzen du.

```
function dig_hand(x: Integer) return y: Integer is
Aurre ≡ { x ≥ 0 }
   if x <= 9 then
      y := x;
   else
      w := dig_hand(x/10);
      if w > x mod 10 then
         y := w;
      else
         y := x mod 10;
      end if;
   end if;
Post ≡ { y = max{x/10^(i-1) mod 10 | i ≥ 1} }
```

$$\textbf{Aurre} \equiv \{\ x \geq 0\ \}$$

$$\textbf{Post} \equiv \{\ y = max\{\tfrac{x}{10^{i-1}}\ mod\ 10 \mid i \geq 1\}\ \}$$

*Soluzioa*:

Frogatu behar dugu hurrengo baieztapena kasu induktiboan betetzen dela:

$$\{\ x \geq 0\ \}\ \texttt{y := dig\_hand(x);}\ \{\ y = max\{\tfrac{x}{10^{i-1}}\ mod\ 10 \mid i \geq 1\}\ \}$$

Kasu induktiboa $x > 9$ da eta erabili behar den indukzio hipotesia:

**(I.H.)** $\{\ \frac{x}{10} \geq 0\ \}$
`w := dig_hand(x/10)`
$\{\ w = max\{\frac{\frac{x}{10}}{10^{i-1}}\ mod\ 10 \mid i \geq 1\}\ \}$
$\equiv \{\ w = max\{\frac{x}{10^{i}}\ mod\ 10 \mid i \geq 1\}\ \}$

Frogapena:

1. $(\ x \geq 0\ \wedge\ x > 9\ )\ \rightarrow\ (\ x > 9\ )\ \rightarrow\ (\ \frac{x}{10} \geq 0\ )$

2. $\{\ \frac{x}{10} \geq 0\ \}$
   `w := dig_hand(x/10);`
   $\{\ w = max\{\frac{x}{10^{i}}\ mod\ 10 \mid i \geq 1\}\ \}$ **(I.H.)**

3. $\{\ x \geq 0\ \wedge\ x > 9\ \}$
   `w := dig_hand(x/10);`
   $\{\ w = max\{\frac{x}{10^{i}}\ mod\ 10 \mid i \geq 1\}\ \}$ 1, 2 eta **(ODE)**

4. $(\ w = max\{\frac{x}{10^{i}}\ mod\ 10 \mid i \geq 1\}\ \wedge\ w > x\ mod\ 10\ )$
   $\rightarrow\ (\ w = max\{\frac{x}{10^{i-1}}\ mod\ 10 \mid i \geq 1\}\ )$

5. $\{\ w = max\{\frac{x}{10^{i-1}}\ mod\ 10 \mid i \geq 1\}\ \}$
   `y := w;`
   $\{\ y = max\{\frac{x}{10^{i-1}}\ mod\ 10 \mid i \geq 1\}\ \}$ **(AA)**

6. $\{\ w = max\{\frac{x}{10^{i}}\ mod\ 10 \mid i \geq 1\}\ \wedge\ w > x\ mod\ 10\ \}$
   `y := w;`
   $\{\ y = max\{\frac{x}{10^{i-1}}\ mod\ 10 \mid i \geq 1\}\ \}$ 4, 5 eta **(ODE)**

7. $(\ w = max\{\frac{x}{10^i}\ mod\ 10 \mid i \geq 1\}\ \wedge\ w \leq x\ mod\ 10\ )$
$\rightarrow\ (\ x\ mod\ 10 = max\{\frac{x}{10^{i-1}}\ mod\ 10 \mid i \geq 1\}\ )$

8. $\{\ x\ mod\ 10 = max\{\frac{x}{10^{i-1}}\ mod\ 10 \mid i \geq 1\}\ \}$
```
y := x mod 10;
```
$\{\ y = max\{\frac{x}{10^{i-1}}\ mod\ 10 \mid i \geq 1\}\ \}$ **(AA)**

9. $\{\ w = max\{\frac{x}{10^i}\ mod\ 10 \mid i \geq 1\}\ \wedge\ w \leq x\ mod\ 10\ \}$
```
y := x mod 10;
```
$\{\ y = max\{\frac{x}{10^{i-1}}\ mod\ 10 \mid i \geq 1\}\ \}$ 7, 8 eta **(ODE)**

10. $(\ x \geq 0\ \wedge\ x > 9\ )\ \rightarrow\ def(w > x\ mod\ 10)$

11. $\{\ w = max\{\frac{x}{10^i}\ mod\ 10 \mid i \geq 1\}\ \}$
```
if w > x mod 10 then
   y := w;
elsif y := x mod 10; then
```
$\{\ y = max\{\frac{x}{10^{i-1}}\ mod\ 10 \mid i \geq 1\}\ \}$ 6, 9, 10 eta **(BDE)**

12. $\{\ x \geq 0\ \wedge\ x > 9\ \}$
```
w := dig_hand(x/10);
if w > x mod 10 then
   y := w;
elsif y := x mod 10; then
```
$\{\ y = max\{\frac{x}{10^{i-1}}\ mod\ 10 \mid i \geq 1\}\ \}$ 3, 11 eta **(KPE)**

3.2. *konb* funtzioak $m$ eta $n$-ren konbinatoria kalkulatzen du. Kontuan izan errekurtsibitateak aukera ematen digula faktorialak kalkulatzean sortzen diren zenbaki handien kalkulua ebitatzeko, tarteko balioak erabiliz.

```
function konb(m,n:  Integer) return r:  Integer is
```
Aurre $\equiv \{\ m \geq n \geq 0\ \}$
```
   r1,r2:  Integer;
   if m = n or n = 0 then
      r := 1;
   else
      r1 := konb(m-1,n);
      r2 := konb(m-1,n-1);
      r := r1+r2;
   end if;
```
Post $\equiv \{\ r = \frac{m!}{n! * (m-n)!}\ \}$

*Soluzioa*:

Frogatu behar da hurrengo baieztapena betetzen dela kasu induktiboan:

$\{\ m \geq n \geq 0\ \}$ `r := konb(m,n);` $\{\ r = \frac{m!}{n! * (m-n)!}\ \}$

Kasu induktiboa $m \neq n\ \wedge\ n \neq 0$ da eta, bi dei errekurtsibo erabiltzen direnez, bi indukzio hipotesi erabili behar dira:

**(I.H. 1)** $\{ \; m-1 \geq n \geq 0 \; \}$

```
r1 := konb(m-1,n);
```

$\{ \; r_1 = \frac{(m-1)!}{n!*((m-1)-n)!} \; \}$

**(I.H. 2)** $\{ \; m-1 \geq n-1 \geq 0 \; \}$

```
r2 := konb(m-1,n-1);
```

$\{ \; r_2 = \frac{(m-1)!}{(n-1)!*((m-1)-(n-1))!} \; \}$

Frogapena:

1. $( \; m \geq n \geq 0 \; \wedge \; m \neq n \; \wedge \; n \neq 0 \; ) \; \rightarrow \; ( \; m > n > 0 \; )$

2. $\{ \; m > n > 0 \; \}$
   `r1 := konb(m-1,n);`
   $\{ \; m > n > 0 \; \wedge \; r_1 = \frac{(m-1)!}{n!*((m-1)-n)!} \; \}$ **(I.H. 1)**

3. $\{ \; m \geq n \geq 0 \; \wedge \; m \neq n \; \wedge \; n \neq 0 \; \}$
   `r1 := konb(m-1,n);`
   $\{ \; m > n > 0 \; \wedge \; r_1 = \frac{(m-1)!}{n!*((m-1)-n)!} \; \}$ 1, 2 eta **(ODE)**

4. $\{ \; m > n > 0 \; \wedge \; r_1 = \frac{(m-1)!}{n!*((m-1)-n)!} \; \}$
   `r2 := konb(m-1,n-1);`
   $\{ \; r_1 = \frac{(m-1)!}{n!*((m-1)-n)!} \; \wedge r_2 = \frac{(m-1)!}{(n-1)!*((m-1)-(n-1))!} \; \}$ **(I.H. 2)**

5. $( \; r_1 = \frac{(m-1)!}{n!*((m-1)-n)!} \; \wedge r_2 = \frac{(m-1)!}{(n-1)!*((m-1)-(n-1))!} \; )$
   $\rightarrow \; ( \; r_1 + r_2 = \frac{(m-1)!}{n!*((m-1)-n)!} + \frac{(m-1)!}{(n-1)!*((m-1)-(n-1))!} \; )$
   $\rightarrow \; ( \; r_1 + r_2 = \frac{(m-1)!}{n!*((m-1)-n)!} + \frac{(m-1)!}{(n-1)!*(m-n)!} \; ) \; )$
   $\rightarrow \; ( \; r_1 + r_2 = \frac{(m-1)!*(m-n)}{n!*((m-n)!} + \frac{(m-1)!*n}{(n)!*(m-n)!} \; )$
   $\rightarrow \; ( \; r_1 + r_2 = \frac{(m-1)!\times(m-n)+(m-1)!\times n}{n!*(m-n)!} \; )$
   $\rightarrow \; ( \; r_1 + r_2 = \frac{(m-1)!\times((m-n)+n)}{n!*(m-n)!} \; )$
   $\rightarrow \; ( \; r_1 + r_2 = \frac{(m-1)!\times m}{n!*(m-n)!} \; ) \; \rightarrow \; ( \; r_1 + r_2 = \frac{m!}{n!*(m-n)!} \; )$

6. $\{ \; r_1 + r_2 = \frac{m!}{n!*(m-n)!} \; \}$
   `r := r1+r2;`
   $\{ \; r = \frac{m!}{n!*(m-n)!} \; \}$ **(AA)**

7. $\{ \; r_1 = \frac{(m-1)!}{n!*((m-1)-n)!} \; \wedge r_2 = \frac{(m-1)!}{(n-1)!*((m-1)-(n-1))!} \; \}$
   `r := r1+r2;`
   $\{ \; r = \frac{m!}{n!*(m-n)!} \; \}$ 5, 6 eta **(ODE)**

8. $\{ \; m \geq n \geq 0 \; \wedge \; m \neq n \; \wedge \; n \neq 0 \; \}$
   `r1 := konb(m-1,n);`
   `r2 := konb(m-1,n-1);`
   `r := r1+r2;`
   $\{ \; r = \frac{m!}{n!*(m-n)!} \; \}$ 3, 4, 7 eta **(KPE)**

4. Idatzi honako programa errekurtsibo hauen aurre-ondoetako espezifikazioa, eta zuzenak direla egiaztatu.

4.1. $x$ eta $b$ ($b > 1$) zenbaki osoko positiboak emanda, $log(b, x)$ funtzioak $b$ oinarriko $x$-ren logaritmoaren zati osoa kalkulatzen du
(Oharra: ( $log_b\ x = z$ ) $\leftrightarrow$ ( $b^z = x$ ) ).

```
function log(b,x:  Integer) return y:  Integer is
   if x = 1 or x < b then
      y := 0;
   else
      y := log(b,x/b);
      y := y+1;
   end if;
```

*Soluzioa*: Frogatu behar den baieztapena:

$\{ x > 0 \land b > 1 \}$ `y := log(b,x);` $\{ x \geq b^y \land x < b^{y+1} \}$

- Kasu nabaria: $x = 1 \lor x < b$

1. $( x > 0 \land b > 1 \land ( x = 1 \lor x < b ) )$
   $\rightarrow ( ( x > 0 \land b > 1 \land x = 1 ) \lor ( x > 0 \land b > 1 \land x < b ) )$
   $\rightarrow ( ( b > 1 \land x = 1 ) \lor ( x > 0 \land b > 1 \land x < b ) )$
   $\rightarrow ( ( x = b^0 \land b > 1 ) \lor ( x \geq b^0 \land x < b^1 ) )$
   $\rightarrow ( x \geq b^0 \land x < b^{0+1} )$

2. $\{ x \geq b^0 \land x < b^{0+1} \}$
   `y := 0;`
   $\{ x \geq b^y \land x < b^{y+1} \}$ **(AA)**

3. $\{ x > 0 \land b > 1 \land ( x = 1 \lor x < b ) \}$
   `y := 0;`
   $\{ x \geq b^y \land x < b^{y+1} \}$ 1, 2 eta **(ODE)**

- Kasu induktiboa: $x \neq 1 \land x \geq b$

**(I.H.)** $\{ \frac{x}{b} > 0 \land b > 1 \}$ `y := log(b,x/b);` $\{ \frac{x}{b} \geq b^y \land \frac{x}{b} < b^{y+1} \}$

4. $( x > 0 \land b > 1 \land x \neq 1 \land x \geq b )$
   $\rightarrow ( \frac{x}{b} > 0 \land b > 1 )$

5. $\{ \frac{x}{b} > 0 \land b > 1 \}$
   `y := log(b,x/b);`
   $\{ \frac{x}{b} \geq b^y \land \frac{x}{b} < b^{y+1} \}$ **(I.H.)**

6. $\{ x > 0 \land b > 1 \land x \neq 1 \land x \geq b \}$
   `y := log(b,x/b);`
   $\{ \frac{x}{b} \geq b^y \land \frac{x}{b} < b^{y+1} \}$ 4, 5 eta **(ODE)**

7. $\{ \frac{x}{b} \geq b^y \land \frac{x}{b} < b^{y+1} \}$
   `y := y+1;`
   $\{ \frac{x}{b} \geq b^{y-1} \land \frac{x}{b} < b^y \}$ **(AA)**

8. $( \frac{x}{b} \geq b^{y-1} \;\wedge\; \frac{x}{b} < b^{y} )$
$\rightarrow ( x \geq b \times b^{y-1} \;\wedge\; x < b \times b^{y} )$
$\rightarrow ( x \geq b^{y} \;\wedge\; x < b^{y+1} )$

9. $\{ \frac{x}{b} \geq b^{y} \;\wedge\; \frac{x}{b} < b^{y+1} \}$
`y := y+1;`
$\{ x \geq b^{y} \;\wedge\; x < b^{y+1} \}$ 7, 8 eta **(ODE)**

10. $\{ x > 0 \;\wedge\; b > 1 \;\wedge\; x \neq 1 \;\wedge\; x \geq b \}$
`y := log(b,x/b);`
`y := y+1;`
$\{ x \geq b^{y} \;\wedge\; x < b^{y+1} \}$ 6, 9 eta **(KPE)**

- Balidazioa: $E \equiv x$
  - Kasu nabaria: $x = 1 \vee x < b$

$$( x > 0 \;\wedge\; b > 1 \;\wedge\; ( x = 1 \;\vee\; x < b ) ) \rightarrow ( x = 1 \;\vee\; ( 0 < x < b ) ) \rightarrow x \in \mathbb{N}$$

  - Kasu induktiboa: $x \neq 1 \wedge x \geq b$

$$( x > 0 \;\wedge\; b > 1 \;\wedge\; x \neq 1 \;\wedge\; x \geq b ) \rightarrow ( x \geq b > 1 ) \rightarrow ( x \in \mathbb{N} \;\wedge\; \frac{x}{b} \in \mathbb{N} \;\wedge\; x > \frac{x}{b} )$$

4.2. *exp* funtzioak berredura kalkulatzen du.[1]

```
function exp(x,y: Integer) return r: Integer is
Aurre ≡ { x > 0 ∧ y ≥ 0 }
   m: Integer;
   if y = 0 then
      r := 1;
   elsif even(y) then
      m := exp(x,y/2);
      r := m*m;
   else
      m := exp(x,y/2);
      r := m*m*x;
   end if;
```

*Soluzioa*: Frogatu behar den baieztapena:

$$\{ x > 0 \;\wedge\; y \geq 0 \} \;\; \texttt{r := exp(x,y);} \;\; \{ r = x^{y} \}$$

- Kasu nabaria: $y = 0$

1. $( x > 0 \;\wedge\; y \geq 0 \;\wedge\; y = 0 )$
$\rightarrow ( 1 = x^{y} )$

[1] $even(y) \equiv y \; mod \; 2 = 0$.

2. $\{ 1 = x^y \}$

```
r := 1;
```

$\{ r = x^y \}$ **(AA)**

3. $\{ x > 0 \wedge y \geq 0 \wedge y = 0 \}$

```
r := 1;
```

$\{ r := x^y \}$ 1, 2 eta **(ODE)**

- 1. kasu induktiboa: $y \neq 0 \wedge even(y)$

**(I.H.)** $\{ x > 0 \wedge \frac{y}{2} \geq 0 \}$ `m := exp(x,y/2);` $\{ m = x^{\frac{y}{2}} \}$

4. $( x > 0 \wedge y \geq 0 \wedge x \neq y \wedge even(y)) \rightarrow ( x > 0 \wedge \frac{y}{2} \geq 0 \wedge even(y) )$

5. $\{ x > 0 \wedge \frac{y}{2} \geq 0 \wedge even(y) \}$

```
m := exp(x,y/2);
```

$\{ m = x^{\frac{y}{2}} \wedge even(y) \}$ **(I.H.)**

6. $\{ x > 0 \wedge y \geq 0 \wedge x \neq y \wedge even(y) \}$

```
m := exp(x,y/2);
```

$\{ m = x^{\frac{y}{2}} \wedge even(y) \}$ 4, 5 eta **(ODE)**

7. $( m = x^{\frac{y}{2}} \wedge even(y) ) \rightarrow ( m = x^{\frac{y}{2}} \wedge x^{\frac{y}{2}} \times x^{\frac{y}{2}} = x^y ) \rightarrow ( m \times m = x^y )$

8. $\{ m \times m = x^y \}$

```
r := m*m;
```

$\{ r = x^y \}$ **(AA)**

9. $\{ m = x^{\frac{y}{2}} \wedge even(y) \}$

```
r := m*m;
```

$\{ r = x^y \}$ 7, 8 eta **(ODE)**

10. $\{ x > 0 \wedge y \geq 0 \wedge x \neq y \wedge even(y) \}$

```
m := exp(x,y/2);
r := m*m;
```

$\{ r = x^y \}$ 6, 9 eta **(KPE)**

- 2. kasu induktiboa: $y \neq 0 \wedge \neg even(y)$

**(I.H.)** $\{ x > 0 \wedge \frac{y}{2} \geq 0 \}$ `m := exp(x,y/2);` $\{ m = x^{\frac{y}{2}} \}$

11. $( x > 0 \wedge y \geq 0 \wedge x \neq y \wedge \neg even(y) ) \rightarrow ( x > 0 \wedge \frac{y}{2} \geq 0 \wedge \neg even(y) )$

12. $\{ x > 0 \wedge \frac{y}{2} \geq 0 \wedge \neg even(y) \}$

```
m := exp(x,y/2);
```

$\{ m = x^{\frac{y}{2}} \wedge \neg even(y) \}$ **(I.H.)**

13. $\{ x > 0 \wedge y \geq 0 \wedge x \neq y \wedge \neg even(y) \}$

```
m := exp(x,y/2);
```

$\{ m = x^{\frac{y}{2}} \wedge \neg even(y) \}$ 11, 12 eta **(ODE)**

14. $( \, m = x^{\frac{y}{2}} \; \wedge \; \neg even(y) \, ) \; \rightarrow \; ( \, m = x^{\frac{y}{2}} \; \wedge \; x^{\frac{y}{2}} \times x^{\frac{y}{2}} \times x = x^{y} \, )$
$\rightarrow \; ( \, m \times m \times x = x^{y} \, )$

15. $\{ \, m \times m \times x = x^{y} \, \}$
```
r := m*m*x;
```
$\{ \, r = x^{y} \, \}$ **(AA)**

16. $\{ \, m = x^{\frac{y}{2}} \; \wedge \; \neg even(y) \, \}$
```
r := m*m*x;
```
$\{ \, r = x^{y} \, \}$ 14, 15 eta **(ODE)**

17. $\{ \, x > 0 \; \wedge \; y \geq 0 \; \wedge \; x \neq y \; \wedge \; \neg even(y) \, \}$
```
m := exp(x,y/2);
r := m*m*x;
```
$\{ \, r = x^{y} \, \}$ 13, 16 eta **(KPE)**

- Balidazioa: $E \; \equiv \; y$
  - Kasu nabaria: $y = 0$

$$( \, x > 0 \; \wedge \; y \geq 0 \; \wedge \; y = 0 \, ) \; \rightarrow \; ( \, y = 0 \, ) \; \rightarrow \; y \in \mathbb{N}$$

  - 1. kasu induktiboa: $even(y)$:

$$( \, x > 0 \; \wedge \; y \geq 0 \; \wedge \; even(y) \, ) \; \rightarrow \; ( \, \tfrac{y}{2} \geq 0 \; \wedge even(y) \, )$$
$$\rightarrow \; ( \, \tfrac{y}{2} \in \mathbb{N} \; \wedge \; y > \tfrac{y}{2} \, )$$

  - 2. kasu induktiboa: $\neg even(y)$:

$$( \, x > 0 \; \wedge \; y \geq 0 \; \wedge \; \neg even(y) \, ) \; \rightarrow \; ( \, \tfrac{y}{2} \geq 0 \; \wedge \neg even(y) \, )$$
$$\rightarrow \; ( \, \tfrac{y}{2} \in \mathbb{N} \; \wedge \; y > \tfrac{y}{2} \, )$$