Duality. Exercises

1. Write the dual models of the following linear problems:

1.1.
$$\begin{aligned} \min\ z = 2x_1 + 3x_2 - 4x_3 \\ \text{subject to} \\ x_1 + 2x_2 + 5x_3 \geq 1 \\ 2x_1 - 2x_2 + 4x_3 = 7 \\ x_1 + 2x_2 + x_3 \geq 10 \\ x_1 \leq 0, x_2 \geq 0, x_3 : \text{unrestricted} \end{aligned}$$

1.2.
$$\begin{aligned} \min\ z = x_1 + 3x_2 + x_3 \\ \text{subject to} \\ 4x_1 - x_2 + 2x_3 \leq -7 \\ 2x_1 - 4x_2 \geq 12 \\ 2x_1 + 8x_2 + 4x_3 \geq 5 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

1.3.
$$\begin{aligned} \max\ z = 2x_1 + 2x_2 + 5x_3 \\ \text{subject to} \\ 2x_1 + x_2 + 2x_3 = 12 \\ -x_1 + 5x_2 - 2x_3 \geq -8 \\ 3x_1 + 4x_2 - 6x_3 \leq 10 \\ x_1 \leq 0, x_2, x_3 \geq 0 \end{aligned}$$

1.4.
$$\begin{aligned} \max\ z = x_1 + x_2 + 5x_3 \\ \text{subject to} \\ x_1 + x_2 + 2x_3 \leq -4 \\ -x_1 + 6x_2 + 2x_3 \geq 2 \\ 4x_1 - x_2 + x_3 = 6 \\ x_1, x_2 \geq 0, x_3 : \text{unrestricted} \end{aligned}$$

1.5.
$$\begin{aligned} \min\ z = 4x_1 + x_2 - x_3 + 2x_4 \\ \text{subject to} \\ 4x_1 - 2x_2 + 3x_3 + x_4 \leq -6 \\ x_1 + x_2 + x_3 + x_4 = 6 \\ 5x_1 + 2x_2 - x_3 - x_4 \geq 10 \\ x_1, x_2 \leq 0, x_3, x_4 \geq 0 \end{aligned}$$

1.6.
$$\begin{aligned} \max\ z = x_1 + 4x_2 \\ \text{subject to} \\ 2x_1 - 4x_2 \leq 14 \\ -x_1 + 8x_2 \geq -6 \\ 4x_1 + 6x_2 \leq 10 \\ x_1 + 9x_2 = 3 \\ x_1 \geq 0, x_2 \leq 0 \end{aligned}$$

2. Consider the following linear models. Write the corresponding dual models, and solve both of them using the graphical solution. What type of solution do they have? A unique solution, multiple solutions, the problem is unbounded or the problem is infeasible.

2.1. $\min\ z = 4x_1 + 6x_2$

subject to

$$\begin{aligned} 2x_1 + x_2 &\geq 4 \\ x_1 + 4x_2 &\geq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

2.2. $\max\ z = 4x_1 + 6x_2$

subject to

$$\begin{aligned} 10x_1 + 12x_2 &\leq 22 \\ 2x_1 + 6x_2 &\leq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

2.3. $\max\ z = -2x_1 + 6x_2$

subject to

$$\begin{aligned} -x_1 + 3x_2 &\leq 9 \\ x_1 + x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

2.4. $\max\ z = -3x_1 + 2x_2$

subject to

$$\begin{aligned} -4x_1 + 2x_2 &\geq 2 \\ x_1 - 2x_2 &\leq -4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

3. Solve the following linear models applying the dual simplex algorithm.

3.1. $\max\ z = -2x_1 - 4x_2 - 3x_3$

subject to

$$\begin{aligned} 2x_1 + x_2 + 2x_3 &\geq 8 \\ 4x_1 + 2x_2 + 2x_3 &\geq 10 \\ 6x_1 + x_2 + 4x_3 &\geq 12 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

3.2. $\min\ z = 2x_1 + x_2 + 3x_3 + 2x_4$

subject to

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 + 2x_4 &\geq 22 \\ 4x_1 + 4x_2 + x_3 + 4x_4 &\leq 20 \\ 2x_1 + 8x_2 + 2x_3 + x_4 &\geq 15 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

3.3. $\max\ z = -2x_1 - 3x_2 - x_3 - x_4$

subject to

$$\begin{aligned} x_1 + x_2 + 3x_3 + x_4 &\leq 40 \\ 2x_1 + 3x_2 + x_3 + x_4 &\geq 30 \\ 2x_1 \qquad\quad + x_3 \qquad\quad &\leq 25 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

3.4. $\max\ z = -6x_1 - 4x_2 - 5x_3 - 4x_4$

subject to

$$\begin{aligned} 2x_1 + 4x_2 + 2x_3 + 5x_4 &\leq 10 \\ x_1 + 2x_2 \qquad\qquad + x_4 &\geq 25 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

3.5. $$\begin{aligned} \max \ z = -2x_1 - x_2 - 2x_3 - x_4 \\ \text{subject to} \\ 6x_1 + 2x_2 + 6x_3 + 3x_4 \leq 12 \\ 2x_1 + x_2 + 2x_3 + 2x_4 \geq 12 \\ x_1 + 2x_2 + 6x_3 + 4x_4 \geq 14 \\ x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

3.6. $$\begin{aligned} \max \ z = -3x_1 + 4x_2 + 2x_3 + 5x_4 \\ \text{subject to} \\ 4x_1 + 2x_2 + 4x_3 + 3x_4 \leq 48 \\ -x_1 + 2x_2 - x_3 + 2x_4 \geq 8 \\ 2x_1 - x_2 + x_3 + x_4 \geq 6 \\ x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

3.7. $$\begin{aligned} \max \ z = 3x_1 - 2x_2 + 2x_3 + x_4 \\ \text{subject to} \\ 3x_1 + 6x_2 + 3x_3 + 2x_4 \leq 36 \\ x_1 + 2x_2 + 3x_3 + x_4 \geq 14 \\ x_1 + x_2 + x_3 + 2x_4 \geq 10 \\ x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

3.8. $$\begin{aligned} \max \ z = 6x_1 + 5x_2 + 5x_3 \\ \text{subject to} \\ -x_1 + x_2 + 2x_3 \geq 40 \\ 2x_1 - 2x_2 - x_3 \geq 30 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

3.9. $$\begin{aligned} \max \ z = 4x_1 - 2x_2 + 3x_3 - 3x_4 \\ \text{subject to} \\ 6x_1 - 6x_2 + 9x_3 + 3x_4 \geq 28 \\ 3x_1 + x_2 + x_3 - 3x_4 \geq 22 \\ x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

4. Consider the following linear model:

$$\begin{aligned} \max \ z = 10x_1 + 6x_2 \\ \text{subject to} \\ x_1 + 2x_2 \leq 2 \\ 2x_1 + x_2 \leq 3 \\ 2x_1 + 2x_2 \leq 3 \\ 4x_1 + x_2 \leq 2 \\ x_1, x_2 \geq 0 \end{aligned}$$

4.1 Write the corresponding dual model.

4.2 Solve the dual model applying the most appropriate algorithm: the simplex algorithm or the dual simplex algorithm.

4.3 Extract the optimal solution to the primal problem directly from the optimal tableau computed for the dual problem.

5. Consider the following linear model:

$$\begin{aligned} \min\ z = 30x_1 + 28x_2 \\ \text{subject to} \\ 4x_1 + 2x_2 \geq 20 \\ 6x_1 + 4x_2 \geq 16 \\ 4x_1 + 2x_2 \geq 18 \\ 4x_1 + 4x_2 \geq 21 \\ x_1, x_2 \geq 0 \end{aligned}$$

5.1 Write the corresponding dual model.

5.2 Solve the dual model applying the most appropriate algorithm: the simplex algorithm or the dual simplex algorithm.

5.3 Extract the optimal solution to the primal problem directly from the optimal tableau computed for the dual problem.

6. Consider the following linear models and their corresponding optimal tableau:

6.1 The model that follows has been solved using the simplex algorithm.

$$\begin{aligned} \max\ z = 6x_1 + 5x_2 + 4x_3 \\ \text{subject to} \\ 15x_1 + 25x_2 + 30x_3 \leq 90 \\ 15x_1 + 5x_2 + 15x_3 \leq 60 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

	x_1	x_2	x_3	x_4	x_5	
	0	0	$\frac{17}{4}$	$\frac{3}{20}$	$\frac{1}{4}$	$\frac{57}{2}$
$\mathbf{a}_2$	0	1	$\frac{3}{4}$	$\frac{1}{20}$	$-\frac{1}{20}$	$\frac{3}{2}$
$\mathbf{a}_1$	1	0	$\frac{3}{4}$	$-\frac{1}{60}$	$\frac{1}{12}$	$\frac{7}{2}$

6.2 The model that follows has been solved using the simplex algorithm, after adding to the model an artificial variable and penalizing the objective function.

$$\begin{aligned} \max\ z = 2x_1 + x_2 - x_3 \\ \text{subject to} \\ x_1 + 2x_2 + 4x_3 \leq 12 \\ 4x_1 + 2x_2 \qquad \geq 8 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

	x_1	x_2	x_3	x_4	x_5	w_1	
	0	3	9	2	0	M	24
$\mathbf{a}_5$	0	6	16	4	1	-1	40
$\mathbf{a}_1$	1	2	4	1	0	0	12

For each of them, answer the following questions:

(a) Extract the optimal solution to the problem from the optimal tableau.

(b) Write the corresponding dual problem, and extract the optimal solution to the dual problem from the optimal tableau computed for the primal problem.

(c) Interpret the shadow prices.