4. SENSITIVITY ANALYSIS

1. The original problem and the optimal tableau

2. Example

3. Change in vector $\mathbf{b}$

4. Change in vector $\mathbf{c}$

5. Change in a nonbasic vector $\mathbf{a}_j$

6. The addition of a new variable

7. The addition of a new constraint

1. The original problem and the optimal tableau

- **The problem with slack variables**

$$
\begin{aligned}
\max \ z \ &= \mathbf{c}^T\mathbf{x} + \mathbf{0}^T\mathbf{x}_s \\
&\text{subject to} \\
\mathbf{A}\mathbf{x} + \mathbf{I}\mathbf{x}_s &= \mathbf{b} \\
\mathbf{x}, \mathbf{x}_s &\geq \mathbf{0}
\end{aligned}
$$

- **The optimal tableau**

	Original variables ($x_1 \ \ldots \ x_n$)	Slack variables ($x_{n+1} \ \ldots \ x_{n+m}$)	
	$\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}^T$	$\mathbf{c}_B^T\mathbf{B}^{-1}$	$z = \mathbf{c}_B^T\mathbf{x}_B$
$\mathbf{B}$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$

The optimal tableau is primal feasible, because $\mathbf{x}_B \geq 0$. It is also dual feasible, because $z_j - c_j \geq 0$ for any j.

The sensitivity analysis is based on the use of the optimal tableau.

2. Example

A production problem.

Resources	Products A	B	C	Resource availability
1	4	2	3	40
2	2	2	1	30
Benefit	3	2	1	

Let x_1, x_2 and x_3 be the number of units of product A, B and C to be produced.

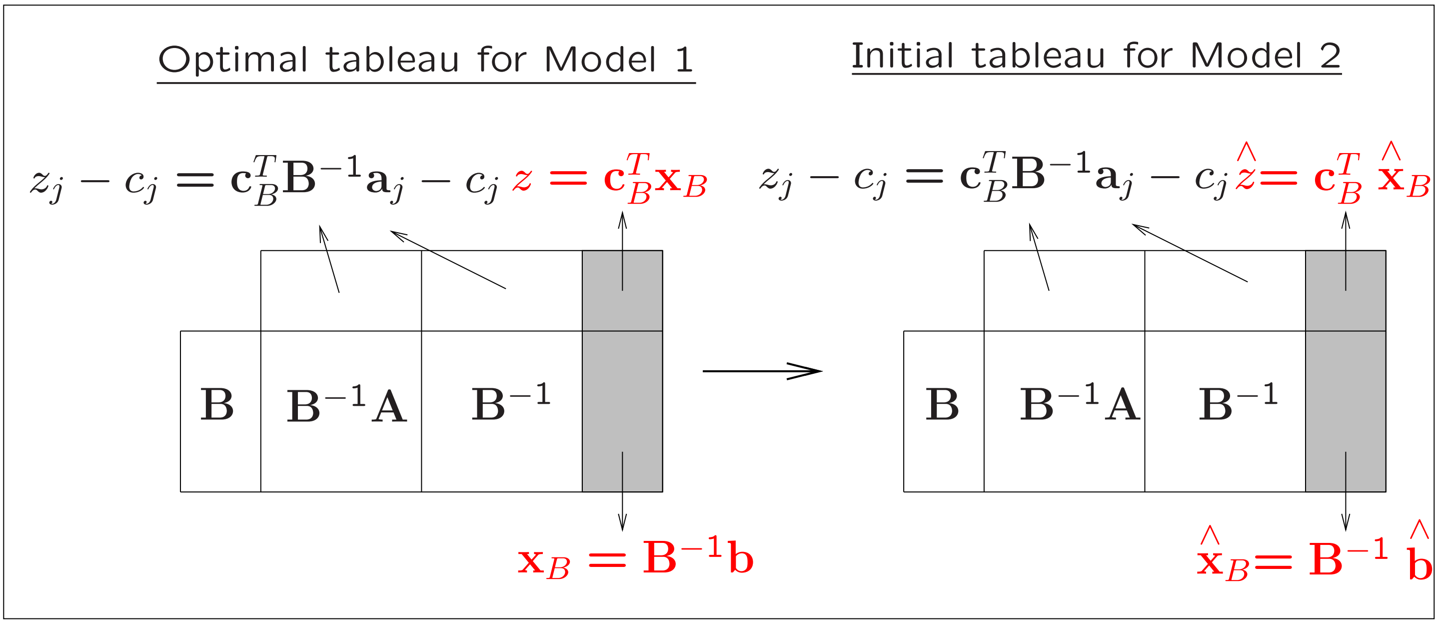
$$\max\ z = 3x_1 + 2x_2 + x_3 + 0x_4 + 0x_5$$

subject to

$$\begin{aligned} 4x_1 + 2x_2 + 3x_3 \quad + x_4 \qquad\quad &= 40 \\ 2x_1 + 2x_2 + x_3 \qquad\qquad + x_5 &= 30 \\ x_1, \ldots, x_5 &\geq 0 \end{aligned}$$

3. Change in vector $\mathbf{b}$

Model 1	Model 2
$\max\ z = \mathbf{c}^T\mathbf{x}$	$\max\ z = \mathbf{c}^T\mathbf{x}$
subject to	subject to
$\mathbf{A}\mathbf{x} \leq \mathbf{b}$	$\mathbf{A}\mathbf{x} \leq \hat{\mathbf{b}}$
$\mathbf{x} \geq \mathbf{0}$	$\mathbf{x} \geq \mathbf{0}$



Two cases:

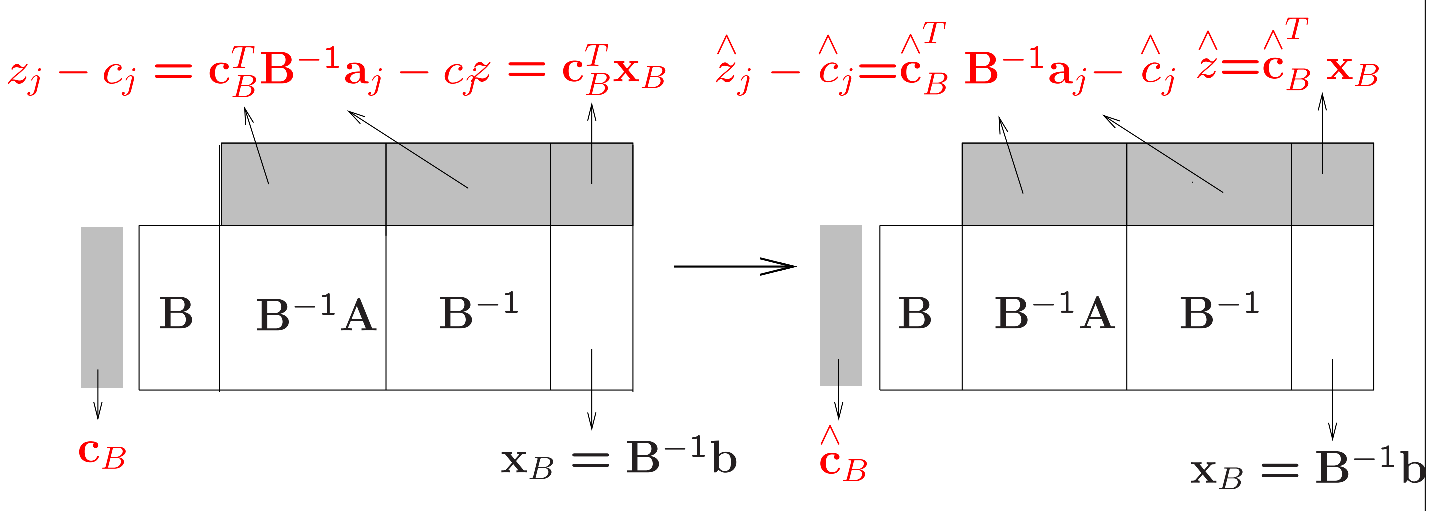
* **Case 1.** If $\hat{\mathbf{x}}_B \geq \mathbf{0}$, B remains optimal and the initial tableau for Model 2 is optimal. The optimal solution: $\hat{\mathbf{x}}_B$. The optimal objective value: $\hat{z}$.

* **Case 2.** If $\hat{\mathbf{x}}_B \ngeq \mathbf{0}$, the tableau is not primal feasible. The dual simplex algorithm will be applied to restore feasibility, starting at the initial tableau for Model 2.

4. Change in vector $\mathbf{c}$

Model 1	Model 2
max $z = \mathbf{c}^T\mathbf{x}$	max $z = \hat{\mathbf{c}}^T\mathbf{x}$
subject to	subject to
$\mathbf{A}\mathbf{x} \leq \mathbf{b}$	$\mathbf{A}\mathbf{x} \leq \mathbf{b}$
$\mathbf{x} \geq \mathbf{0}$	$\mathbf{x} \geq \mathbf{0}$

Optimal tableau for Model 1 — Initial tableau for Model 2



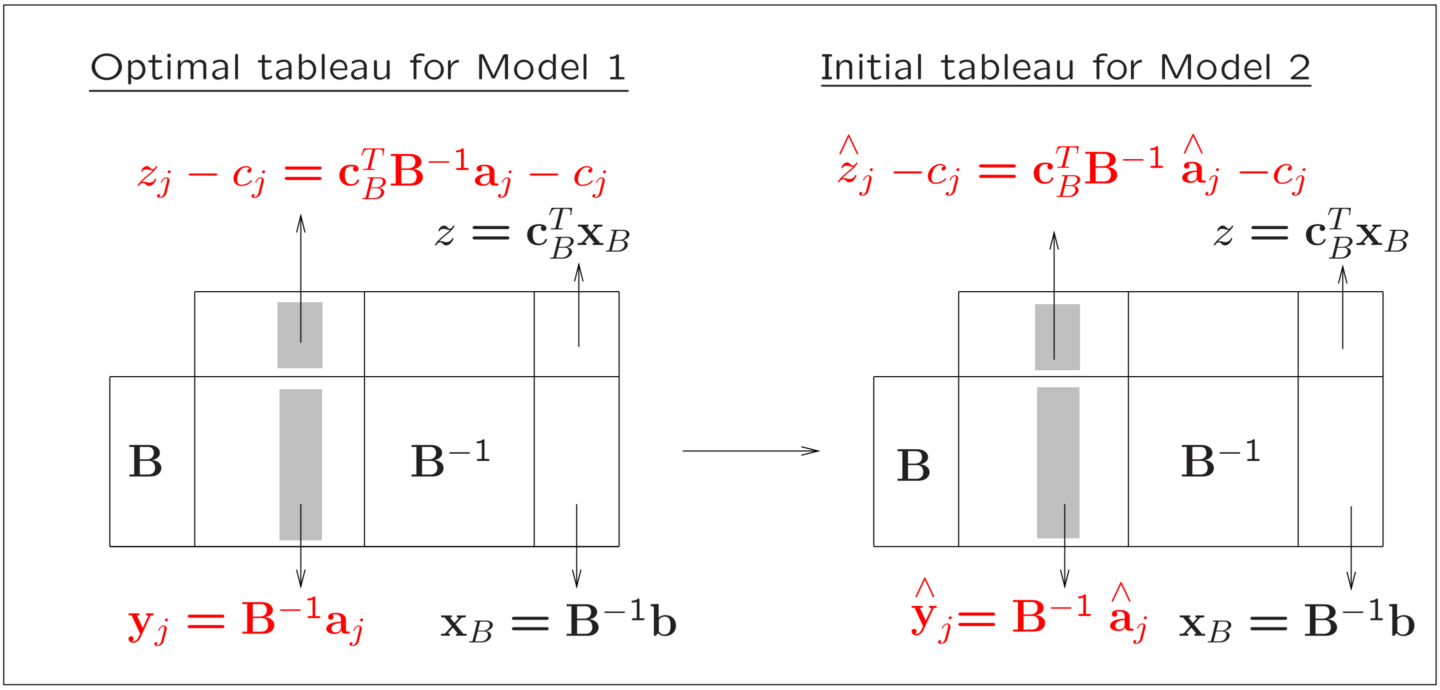
Two cases:

- **Case 1.** If $\hat{z}_j - \hat{c}_j \geq 0$ for any j, $\mathbf{B}$ remains optimal. The initial tableau for Model 2 is optimal; $\mathbf{x}_B$ remains optimal and the optimal objective value is $\hat{z} = \hat{\mathbf{c}}_B^T\mathbf{x}_B$.

- **Case 2.** If there exists any $\hat{z}_j - \hat{c}_j < 0$, the tableau is not dual feasible. The simplex algorithm will be applied to restore feasibility, starting at the initial tableau for Model 2.

5. Change in a nonbasic vector $\mathbf{a}_j$

Model 1	Model 2
max $z = \mathbf{c}^T\mathbf{x}$	max $z = \mathbf{c}^T\mathbf{x}$
subject to	subject to
$\mathbf{a}_1 x_1 + \cdots + \mathbf{a}_j x_j + \cdots + \mathbf{a}_n x_n \leq \mathbf{b}$	$\mathbf{a}_1 x_1 + \cdots + \hat{\mathbf{a}}_j x_j + \cdots + \mathbf{a}_n x_n \leq \mathbf{b}$
$x_1, \cdots, x_n \geq 0$	$x_1, \cdots, x_n \geq 0$



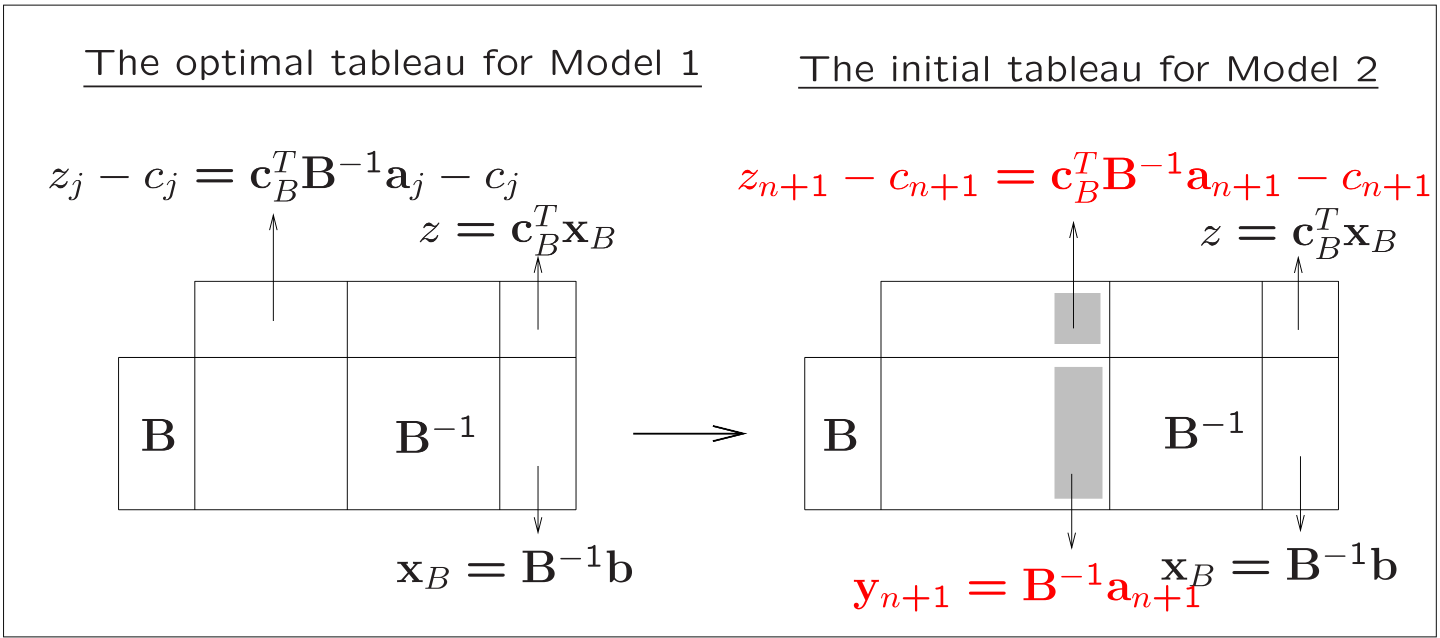
Two cases:

* **Case 1.** If $\hat{z}_j - c_j \geq 0$, the tableau remains dual feasible, and the solution $\mathbf{x}_B$ and the objective value z remain optimal.

* **Case 2.** If $\hat{z}_j - c_j < 0$, the tableau is no longer optimal since it is not dual feasible. The simplex algorithm will be applied to restore feasibility, starting at the initial tableau for Model 2.

6. The addition of a new variable

Model 1	Model 2
max $z = c_1x_1 + \cdots + c_nx_n$	max $z = c_1x_1 + \cdots + c_nx_n + c_{n+1}x_{n+1}$
subject to	subject to
$\mathbf{a}_1x_1 + \cdots + \mathbf{a}_nx_n \leq \mathbf{b}$	$\mathbf{a}_1x_1 + \cdots + \mathbf{a}_nx_n + \mathbf{a}_{n+1}x_{n+1} \leq \mathbf{b}$
$x_1, \cdots, x_n \geq 0$	$x_1, \cdots, x_n, x_{n+1} \geq 0$



Two cases:

* **Case 1.** If the new $z_{n+1} - c_{n+1} \geq 0$, the introduction of a new variable x_{n+1} to the model does not affect the optimal solution, since the initial tableau for the augmented model is dual feasible. The solution $\mathbf{x}_B$ and the objective value z remain optimal.

* **Case 2.** If the new $z_{n+1} - c_{n+1} < 0$, the initial tableau for the augmented model is not optimal since it is not dual feasible. The simplex algorithm will be applied to find the new optimal solution.

7. The addition of a new constraint

Model 1

$$\max\ z = c_1x_1 + \cdots + c_nx_n$$

subject to

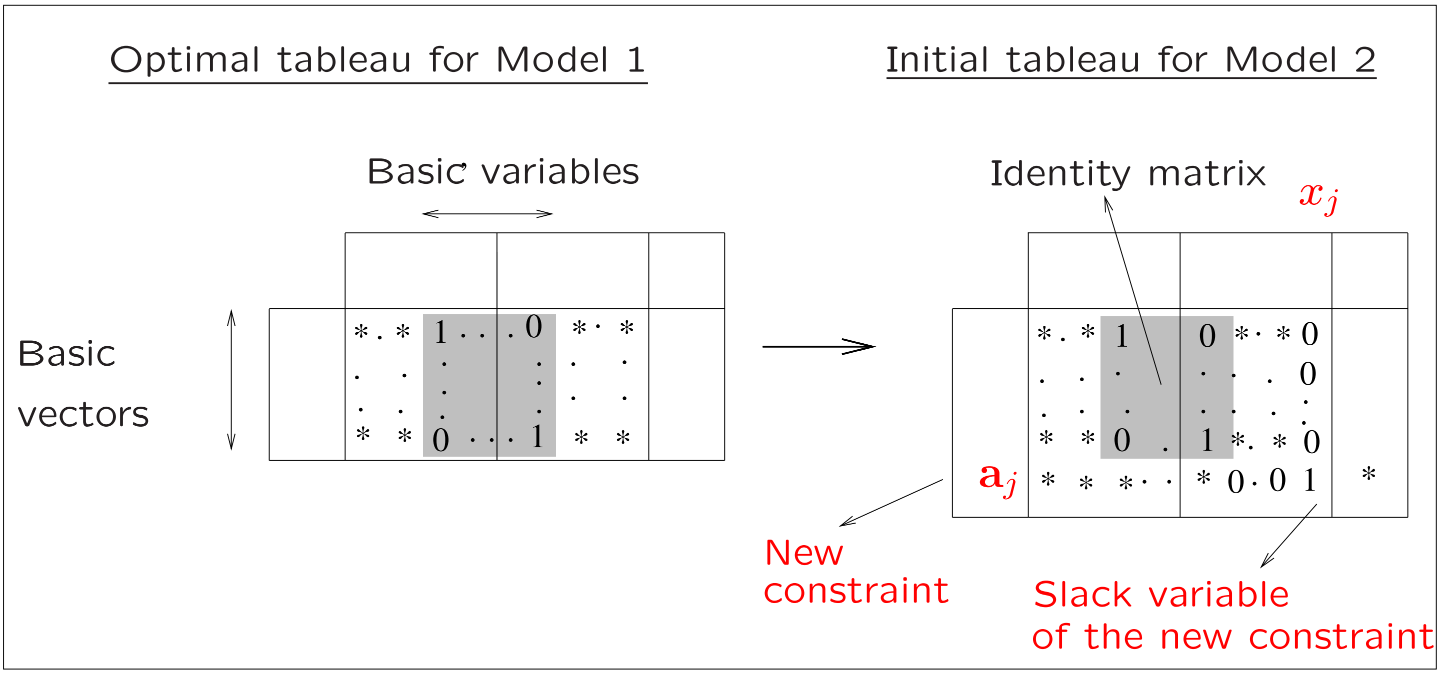
$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &\leq b_1 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &\leq b_m \\ x_1, \cdots, x_n &\geq 0 \end{aligned}$$

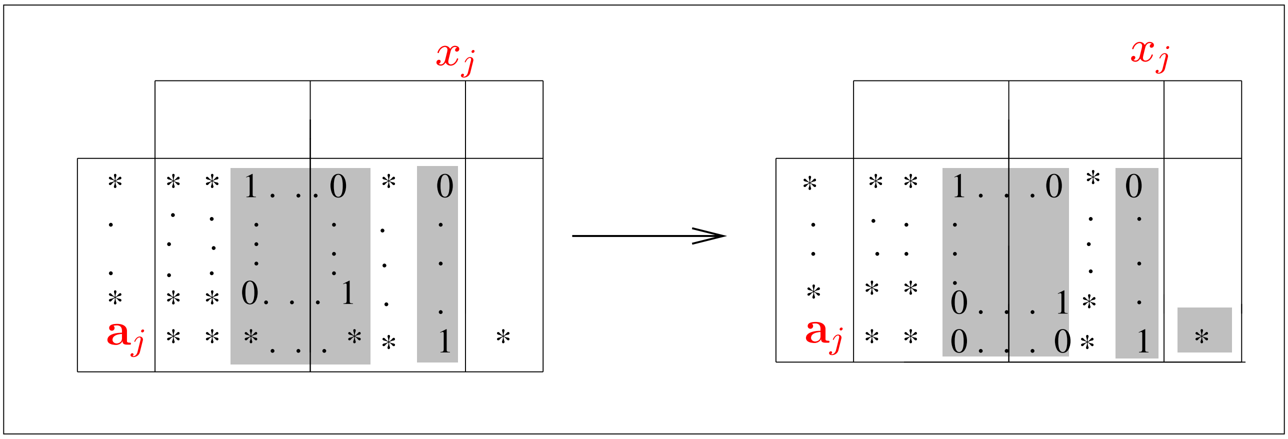
Model 2

$$\max\ z = c_1x_1 + \cdots + c_nx_n$$

subject to

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &\leq b_1 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &\leq b_m \\ a_{m+1,1}x_1 + \cdots + a_{m+1,n}x_n &\leq b_{m+1} \\ x_1, \cdots, x_n &\geq 0 \end{aligned}$$





Two cases:

* **Case 1.** If the initial tableau for the augmented model is primal feasible, then the tableau is optimal. The current optimal solution satisfies the new constraint.

* **Case 2.** If the initial tableau for the augmented model is not primal feasible, then the dual simplex algorithm will be applied to find the optimal solution to Model 2.