EJERCICIOS RESUELTOS

1. Sea el espacio vectorial euclídeo $\left(\mathbb{R}^4, <,>\right)$ con el producto escalar:

$$\forall \overline{x} = (x_1, x_2, x_3, x_4), \overline{y} = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$$

$$< \overline{x}, \overline{y} >= x_1y_1 + x_2y_2 + x_1y_3 + x_3y_1 + 2x_3y_3 + 2x_4y_4$$

a. Hallar una base ortogonal.
b. Hallar una base ortonormal.

SOLUCIÓN 4.1

2. Encontrar la mejor aproximación $\overline{w}$ de $\overline{v} = (1,1,1,1) \in \mathbb{R}^4$ en el subespacio:

$$S = \left\{\left(x_1, x_2, x_3, x_4\right) / \, x_1 - x_4 = 0 \wedge x_2 - x_4 = x_3\right\}$$

Calcular la norma del vector de error.

SOLUCIÓN 4.2

3. Sea la matriz:

$$A = \begin{pmatrix} -5 & 1 & 2 \\ 1 & -5 & 2 \\ 2 & 2 & -2 \end{pmatrix}$$

a) Hallar los valores propios y los subespacios propios de A.
b) Diagonalizar A ortogonalmente.
c) Utilizando el apartado anterior, ¿Cuánto vale Det(A)?.

SOLUCIÓN 4.3

SOLUCIÓN 4.1

Utilizando Mathematica 5.2

```
In[1]:= << LinearAlgebra`Orthogonalization`

In[2]:= pe[x_, y_] := x[[1]] y[[1]] + x[[2]] y[[2]] + x[[1]] y[[3]] +
          x[[3]] y[[1]] + 2 x[[3]] y[[3]] + 2 x[[4]] y[[4]]

In[3]:= b = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};

                                   Base Canónica de R^4

In[4]:= GramSchmidt[b, InnerProduct → pe]

Out[4]= {{1, 0, 0, 0}, {0, 1, 0, 0}, {-1, 0, 1, 0}, {0, 0, 0, 1/√2}}

                                   Base Ortonormal de R^4

In[5]:= GramSchmidt[b, InnerProduct → pe, Normalized → False]

Out[5]= {{1, 0, 0, 0}, {0, 1, 0, 0}, {-1, 0, 1, 0}, {0, 0, 0, 1}}

                                   Base Ortogonal de R^4
```

Utilizando Mathematica 6.0

```
Wolfram Mathematica 6.0 - [Untitled-1.nb *]
File  Edit  Insert  Format  Cell  Graphics  Evaluation  Palettes  Window  Help
Untitled-1.nb *

In[1]:= pe[x_, y_] := x[[1]] y[[1]] + x[[2]] y[[2]] + x[[1]] y[[3]] + x[[3]] y[[1]] +
          2 x[[3]] y[[3]] + 2 x[[4]] y[[4]]

In[2]:= b = {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};

                                   Base Canónica de R^4

In[3]:= Orthogonalize[b, pe]

Out[3]= {{1, 0, 0, 0}, {0, 1, 0, 0}, {-1, 0, 1, 0}, {0, 0, 0, 1/√2}}

                                   Base ortonormal de R^4
```

Con esta versión de Mathematica se obtiene bases ortonormales.

SOLUCIÓN 4.2

Utilizando Mathematica 6.0

In[1]:= `Solve[{`$x_1 - x_4 == 0,\ x_2 - x_4 == x_3$`}]`

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[1]= $\{\{x_1 \to x_2 - x_3,\ x_4 \to x_2 - x_3\}\}$

$S = \{(x_2 - x_3,\ x_2,\ x_3,\ x_2 - x_3) = x_2\,(1, 1, 0, 1) + x_3\,(-1, 0, 1, -1)\} =$
$= L\,((1, 1, 0, 1),\ (-1, 0, 1, -1))$

In[2]:= `u = {{1, 1, 0, 1}, {-1, 0, 1, -1}};`

Comprobamos que el sistema u es libre

In[3]:= `Solve[a1 u[[1]] + a2 u[[2]] == {0, 0, 0, 0}, {a1, a2}]`

Out[3]= `{{a1 → 0, a2 → 0}}`

Ortonormalizamos la base u de S

In[6]:= `e1 = Orthogonalize[u]`

Out[6]= $\left\{\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right\}, \left\{-\frac{1}{\sqrt{15}}, \frac{2}{\sqrt{15}}, \sqrt{\frac{3}{5}}, -\frac{1}{\sqrt{15}}\right\}\right\}$

No ha sido necesario indicar el producto escalar,
pues trabajamos con el producto escalar usual de $\mathbb{R}^4$

In[7]:= `v = {1, 1, 1, 1};`

In[9]:= `vp = Sum[Projection[v, e1[[i]]], {i, 2}]`

Out[9]= $\left\{\frac{4}{5}, \frac{7}{5}, \frac{3}{5}, \frac{4}{5}\right\}$

Mejor aproximación de $\vec{v}$ en S: $vp = \left(\frac{4}{5}, \frac{7}{5}, \frac{3}{5}, \frac{4}{5}\right)$

In[10]:= `e = v - vp`

Out[10]= $\left\{\frac{1}{5}, -\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right\}$

La norma del vector de error

In[11]:= `Norm[e]`

Out[11]= $\sqrt{\frac{2}{5}}$

SOLUCIÓN 4.3

```
In[12]:= a = {{-5, 1, 2}, {1, -5, 2}, {2, 2, -2}};

In[13]:= Eigenvalues[a]

Out[13]= {-6, -6, 0}

In[14]:= Eigenvectors[a]

Out[14]= {{-2, 0, 1}, {-1, 1, 0}, {1, 1, 2}}
```

Valores propios de A:

$$\lambda_1 = -6 \quad k_1 = 2$$
$$\lambda_2 = 0 \quad k_2 = 1$$

Subespacios propios de A:

$$V(-6) = L\left(\left\{(-2,0,1),(-1,1,0)\right\}\right) \qquad d_1 = 2$$
$$V(0) = L\left(\left\{(1,1,2)\right\}\right) \quad d_2 = 1$$

La matriz real y simétrica A es ortogonalmente diagonalizable, existe una matriz Q ortogonal tal que se cumple:

$$D = Q^T AQ$$

```
In[15]:= u = {{-2, 0, 1}, {-1, 1, 0}, {1, 1, 2}};

In[16]:= Orthogonalize[u]
```

$$\text{Out[16]= } \left\{\left\{-\frac{2}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}\right\}, \left\{-\frac{1}{\sqrt{30}}, \sqrt{\frac{5}{6}}, -\sqrt{\frac{2}{15}}\right\}, \left\{\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \sqrt{\frac{2}{3}}\right\}\right\}$$

$$\begin{pmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{30}} & \sqrt{\frac{5}{6}} & -\sqrt{\frac{2}{15}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} -5 & 1 & 2 \\ 1 & -5 & 2 \\ 2 & 2 & -2 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \\ 0 & \sqrt{\frac{5}{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & -\sqrt{\frac{2}{15}} & \sqrt{\frac{2}{3}} \end{pmatrix}$$

$$D = Q^T AQ \quad \to \quad QDQ^T = A \quad \to \quad |D| = |A| = \begin{vmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

```
In[17]:= Det[a]

Out[17]= 0
```