INTEGRAL DEFINIDA

[7.1] Calcular: $\displaystyle\int_1^3 \frac{dx}{x\cdot\sqrt{x^2+5x+1}}$

Solución

$$\int_1^3 \frac{dx}{x\sqrt{x^2+5x+1}} = \left\{\begin{array}{l} x=1/t \quad \Rightarrow \quad dx=-\dfrac{dt}{t^2} \\ \sqrt{x^2+5x+1}=\dfrac{\sqrt{1+5t+t^2}}{t} \end{array}\quad \begin{array}{c|c} x & t \\ \hline 1 & 1 \\ 3 & 1/3 \end{array}\right\} = \int_1^{1/3} \frac{-\dfrac{dt}{t^2}}{\dfrac{1}{t}\cdot\dfrac{\sqrt{t^2+5t+1}}{t}} = \int_{1/3}^1 \frac{dt}{\sqrt{t^2+5t+1}} =$$

$$= \int_{1/3}^1 \frac{dt}{\sqrt{\left(t+\dfrac{5}{2}\right)^2-\dfrac{21}{4}}} = \left[\ln\left|t+\frac{5}{2}+\sqrt{t^2+5t+1}\right|\right]_{1/3}^1 = \ln\left(1+\frac{5}{2}+\sqrt{7}\right)-\ln\left(\frac{1}{3}+\frac{5}{2}+\sqrt{\frac{1}{9}+\frac{5}{3}+1}\right) =$$

$$= \ln\frac{\dfrac{7+2\sqrt{7}}{2}}{\dfrac{1}{3}+\dfrac{5}{2}+\dfrac{5}{3}} = \ln\frac{\dfrac{7+2\sqrt{7}}{2}}{\dfrac{27}{6}} = \ln\frac{7+2\sqrt{7}}{9}$$

Definite integral: More digits

$$\int_1^3 \frac{1}{x\sqrt{x^2+5x+1}}\,dx = \log\left(\frac{1}{9}\left(7+2\sqrt{7}\right)\right) \approx 0.311684\ldots$$

$\log(x)$ is the natural logarithm »

[7.2] Calcular: $\displaystyle\int_0^{\ln 5} \frac{e^x\cdot\sqrt{e^x-1}}{e^x+3}\,dx$

Solución

$$\int_0^{\ln 5} \frac{e^x\cdot\sqrt{e^x-1}}{e^x+3}\,dx = \left[\begin{array}{c} e^x-1=t^2 \to e^x=t^2+1 \to x=\ln(t^2+1) \\ dx=\dfrac{2t}{t^2+1}dt \end{array}\right]\left[\begin{array}{c} x=0 \to e^0-1=t^2 \to t=0 \\ x=\ln 5 \to e^{\ln 5}-1=t^2 \to t=2 \end{array}\right] =$$

$$= \int_0^2 \frac{2t^2}{t^2+4}dt = 2\int_0^2\left(1-\frac{4}{t^2+4}\right)dt = 2\left[t-2\,\mathrm{arctg}\frac{t}{2}\right]_0^2 = 2\left(2-2\frac{\pi}{4}\right) = 4-\pi$$

Definite integral: More digits

$$\int_0^{\log(5)} \frac{e^x \sqrt{e^x - 1}}{e^x + 3}\, dx \approx 0.858407$$

$\log(x)$ is the natural logarithm »

[7.3] Calcular: $\int_0^{\ln 2} \sqrt{e^x - 1}\, dx$

Solución

$$\int_0^{\ln 2} \sqrt{e^x - 1}\, dx = \left\{ \begin{array}{l} e^x - 1 = t^2 \\ x = \ln(1+t^2) \\ dx = \dfrac{2tdt}{1+t^2} \end{array} \quad \begin{array}{c|c} x & t \\ \hline 0 & 0 \\ \ln 2 & 1 \end{array} \right\} = \int_0^1 t \cdot \frac{2t}{t^2+1} dt = 2\int_0^1 \frac{t^2}{t^2+1} dt =$$

$$= 2\int_0^1 \left(1 - \frac{1}{t^2+1}\right) dt = 2\left[t - \operatorname{arctg} t\right]_0^1 = 2(1 - \operatorname{arctg} 1) = 2\left(1 - \frac{\pi}{4}\right) = \boxed{2 - \frac{\pi}{2}}$$

Definite integral: More digits

$$\int_0^{\log(2)} \sqrt{e^x - 1}\, dx = 2 - \frac{\pi}{2} \approx 0.429204\ldots$$

$\log(x)$ is the natural logarithm »

[7.4] Calcular las siguientes integrales:

1) $\int_1^4 \frac{dx}{2\sqrt{x}(1+\sqrt{x})^2}$ 2) $\int_0^1 \frac{dx}{\sqrt{1+x^2}}$

Solución

$$1) \int_1^4 \frac{dx}{2\sqrt{x}(1+\sqrt{x})^2} = \left[\begin{array}{c} 1+\sqrt{x} = t \to \dfrac{1}{2\sqrt{x}} dx = dt \\ x = 1 \to t = 2 \\ x = 4 \to t = 3 \end{array} \right] = \int_2^3 \frac{dt}{t^2} = -\frac{1}{t}\Big]_2^3 = -\left[\frac{1}{3} - \frac{1}{2}\right] = \frac{1}{6}$$

Definite integral: More digits

$$\int_1^4 \frac{1}{2\sqrt{x}\,(1+\sqrt{x})^2}\, dx = \frac{1}{6} \approx 0.166667\ldots$$

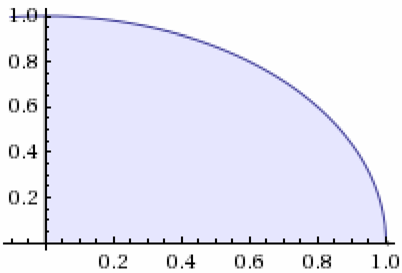
2) $\int_0^1 \sqrt{1-x^2}\,dx = \begin{bmatrix} x=\operatorname{sen} t \rightarrow dx=\cos t\,dt \\ x=0 \rightarrow t=0 \\ x=1 \rightarrow t=\pi/2 \end{bmatrix} = \int_0^{\pi/2} \sqrt{1-\operatorname{sen}^2 t}\cos t\,dt = \int_0^{\pi/2} \cos^2 t\,dt =$

$$= \int_0^{\pi/2} \frac{1}{2}(1+\cos 2t)dt = \frac{1}{2}\left[t+\frac{\operatorname{sen} 2t}{2}\right]_0^{\pi/2} = \frac{\pi}{4}$$

Definite integral: More digits

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4} \approx 0.785398\ldots$$

Visual representation of the integral:



[7.5] Estudiar la convergencia o divergencia de las siguientes integrales impropias

1) $\int_1^{\infty} \frac{1}{x^2}dx$ 2) $\int_1^{\infty} \frac{1}{x}dx$

Solución

1) $$\int_1^{\infty} \frac{1}{x^2}dx = \lim_{u\to\infty}\int_1^{u} \frac{1}{x^2}dx = \lim_{u\to\infty}\left(\left[-\frac{1}{x}\right]_1^u\right) = \lim_{u\to\infty}\left(-\frac{1}{u}+1\right) = 1-0 = 1$$

La integral es convergente.

Definite integral:

$$\int_1^{\infty} \frac{1}{x^2}\,dx = 1$$

2)

$$\int_1^{\infty} \frac{1}{x}dx = \lim_{u\to\infty}\int_1^{u} \frac{1}{x}dx = \lim_{u\to\infty}\left(\left[\ln x\right]_1^u\right) = \lim_{u\to\infty}(\ln u - \ln 1) = \infty - 1 = \infty$$

La integral es divergente.

Input: Mathematica form

$$\int_1^\infty \frac{1}{x}\,dx$$

Result:

(integral does not converge)

[7.6] Estudiar la convergencia o divergencia de las siguientes integrales impropias

1) $\displaystyle\int_{-2}^{0} \frac{1}{(x+1)^3}dx$ 2) $\displaystyle\int_{-1}^{1} \frac{1}{x^4}dx$ 3) $\displaystyle\int_{1}^{3} \frac{1}{\sqrt{x-1}}dx$

Solución

1) La integral es impropia de segunda especie porque el denominador de la función $f(x)=\dfrac{1}{(x+1)^3}$ se anula en $x=-1$.

$$\int_{-2}^{0}\frac{1}{(x+1)^3}dx=\int_{-2}^{-1}\frac{1}{(x+1)^3}dx+\int_{-1}^{0}\frac{1}{(x+1)^3}dx=\lim_{u\to 0^+}\int_{-2}^{-1-u}\frac{1}{(x+1)^3}dx+\lim_{v\to 0^+}\int_{-1+v}^{0}\frac{1}{(x+1)^3}dx=$$

$$=\lim_{u\to 0^+}\left(\left[\frac{-1}{2(x+1)^2}\right]_{-2}^{-1-u}\right)+\lim_{v\to 0^+}\left(\left[\frac{-1}{2(x+1)^2}\right]_{-1+v}^{0}\right)=-\frac{1}{2}\lim_{u\to 0^+}\left(\frac{1}{u^2}-1\right)-\frac{1}{2}\lim_{v\to 0^+}\left(1-\frac{1}{v^2}\right)=$$

$$=-\frac{1}{2}\lim_{u\to 0^+}\frac{1}{u^2}+\frac{1}{2}-\frac{1}{2}+\frac{1}{2}\lim_{v\to 0^+}\frac{1}{v^2}=-\infty+\infty$$

La integral es divergente puesto que no existe el límite anterior, sin embargo, al hacer $u=v$ para obtener el valor principal de Cauchy se simplifican los límites y queda igual a cero.

Input: Mathematica form

$$\int_{-2}^{0}\frac{1}{(x+1)^3}\,dx$$

Result:

(integral does not converge)

Nonintegrable singularity of the integrand in the integration interval:

x	nonintegrable expansion terms of the integrand
-1	$\dfrac{1}{(x+1)^3}$

2) La integral es impropia de segunda especie porque el denominador de la función $f(x)=\dfrac{1}{x^4}$ se anula en $x=0$.

$$\int_{-1}^{1}\frac{1}{x^4}dx=\int_{-1}^{0}\frac{1}{x^4}dx+\int_{0}^{1}\frac{1}{x^4}dx=\lim_{u\to 0^+}\int_{-1}^{-u}\frac{1}{x^4}dx+\lim_{v\to 0^+}\int_{v}^{1}\frac{1}{x^4}dx=$$

$$=\lim_{u\to 0^+}\left(\left]\frac{-1}{3x^3}\right]_{-1}^{-u}\right)+\lim_{v\to 0^+}\left(\left]\frac{-1}{3x^3}\right]_{v}^{1}\right)=\lim_{u\to 0^+}\left(\frac{1}{3u^3}-\frac{1}{3}\right)+\lim_{v\to 0^+}\left(-\frac{1}{3}+\frac{1}{v^2}\right)=$$

$$=-\frac{2}{3}+\lim_{u\to 0^+}\frac{1}{3u^3}+\lim_{v\to 0^+}\frac{1}{3v^3}=\infty$$

La integral es divergente puesto que no existe el límite anterior y en este caso tampoco existe el valor principal de Cauchy.

Input: *Mathematica form*

$$\int_{-1}^{1}\frac{1}{x^4}\,dx$$

Result:

(integral does not converge)

Nonintegrable singularity of the integrand in the integration interval:

x	nonintegrable expansion terms of the integrand
0	$\dfrac{1}{x^4}$

3) Es una integral impropia de segunda especie porque la función $f(x)=\dfrac{1}{\sqrt{x-1}}$ no está acotada en $x=1$

$$\int_{1}^{3}\frac{1}{\sqrt{x-1}}dx=\lim_{u\to 0^+}\int_{1+u}^{3}\frac{1}{\sqrt{x-1}}dx=\lim_{u\to 0^+}2\sqrt{x-1}\Big]_{1+u}^{3}=2\sqrt{2}-\lim_{u\to 0^+}2\sqrt{u}=2\sqrt{2}$$

La integral es convergente

Definite integral: More digits

$$\int_{1}^{3}\frac{1}{\sqrt{x-1}}\,dx=2\sqrt{2}\approx 2.82843\ldots$$

[7.7] Calcular la integral impropia convergente: $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\operatorname{sen} x}}\,dx$

Solución

$$\int \frac{\cos x}{\sqrt{1-\operatorname{sen} x}}\,dx = \begin{bmatrix} 1-\operatorname{sen} x = t^2 \\ -\cos x\,dx = 2t\,dt \end{bmatrix} = -2\int \frac{t}{t}\,dt = -2t + C = -2\sqrt{1-\operatorname{sen} x} + C$$

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\operatorname{sen} x}}\,dx = \lim_{\varepsilon\to 0}\int_0^{\frac{\pi}{2}-\varepsilon} \frac{\cos x}{\sqrt{1-\operatorname{sen} x}}\,dx = \lim_{\varepsilon\to 0}\left[-2\sqrt{1-\operatorname{sen} x}\right]_0^{\frac{\pi}{2}-\varepsilon} = 2$$

Definite integral:

$$\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sqrt{1-\sin(x)}}\,dx = 2$$

[7.8] Estudiar la convergencia de la integral impropia según los valores del parámetro real p:

$$\int_e^{\infty} \frac{1}{x(\ln x)^p}\,dx$$

Solución

- Si $p = 1$:

$$\int_e^{\infty} \frac{1}{x\ln x}\,dx = \begin{bmatrix} \ln x = t & x = e \to t = 1 \\ dx/x = dt & x = \infty \to t = \infty \end{bmatrix} = \int_1^{\infty} (1/t)\,dt = \lim_{u\to\infty} \ln|t|\Big]_1^u = \infty$$

- Si $p \neq 1$:

$$\int_e^{\infty} \frac{1}{x(\ln x)^p}\,dx = \begin{bmatrix} \ln x = t & x = e \to t = 1 \\ dx/x = dt & x = \infty \to t = \infty \end{bmatrix} = \int_1^{\infty} t^{-p}\,dt = \lim_{u\to\infty} \frac{t^{-p+1}}{1-p}\Bigg]_1^u =$$

$$= \begin{cases} \infty & \text{si } p < 1 \\ \dfrac{1}{p-1} & \text{si } p > 1 \end{cases}$$

Es decir:

$$\int_e^{\infty} \frac{1}{x(\ln x)^p}\,dx \text{ converge si } p > 1\text{, diverge si } p \le 1$$

[7.9] Calcular mediante la función beta de Euler: $\displaystyle\int_0^8 \frac{dx}{x^{1/2}\left(2-x^{1/3}\right)^{-1/4}}$

(Indicación: hacer el cambio $x^{1/3}=2t$)

Solución

$$\int_0^8 \frac{dx}{x^{1/2}\left(2-x^{1/3}\right)^{-1/4}} = \left\{\begin{array}{lll} x^{1/3}=2t & x^{2/3}=4t^2 & \begin{array}{c|c} x & t \\ \hline 0 & 0 \\ 8 & 1 \end{array} \\ \frac{1}{3}x^{-2/3}dx=2dt & dx=24t^2dt & \end{array}\right\} =$$

$$=\int_0^1 \frac{24t^2dt}{2\sqrt{2}\,t^{3/2}(2-2t)^{-1/4}} = \frac{12}{\sqrt{2}}\int_0^1 \frac{t^2dt}{t^{3/2}2^{-1/4}(1-t)^{-1/4}} = \frac{12\sqrt[4]{2}}{\sqrt{2}}\int_0^1 t^{1/2}(1-t)^{1/4}dt =$$

$$=\left\{\begin{array}{lll} p-1=\frac{1}{2} & \rightarrow & p=\frac{3}{2} \\ q-1=\frac{1}{4} & \rightarrow & q=\frac{5}{4} \end{array}\right\} = \beta\left(\frac{3}{2},\frac{5}{4}\right) = 6\sqrt[4]{8}\cdot\frac{\Gamma\left(\frac{3}{2}\right)\cdot\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{11}{4}\right)} = 6\sqrt[4]{8}\cdot\frac{\frac{1}{2}\sqrt{\pi}\cdot\Gamma\left(\frac{5}{4}\right)}{\left(\frac{7}{4}\right)\cdot\Gamma\left(\frac{7}{4}\right)} =$$

$$=3\cdot 2^{\frac{3}{4}}\cdot\frac{\sqrt{\pi}\cdot\Gamma\left(\frac{5}{4}\right)}{\left(\frac{7}{4}\right)\cdot\Gamma\left(\frac{7}{4}\right)}$$

Definite integral: More digits

$$\int_0^8 \frac{1}{\sqrt{x}\,\frac{1}{\sqrt[4]{2-\sqrt[3]{x}}}}\,dx = \frac{3\times 2^{3/4}\sqrt{\pi}\,\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{11}{4}\right)} \approx 5.03972\ldots$$

$\Gamma(x)$ is the gamma function »

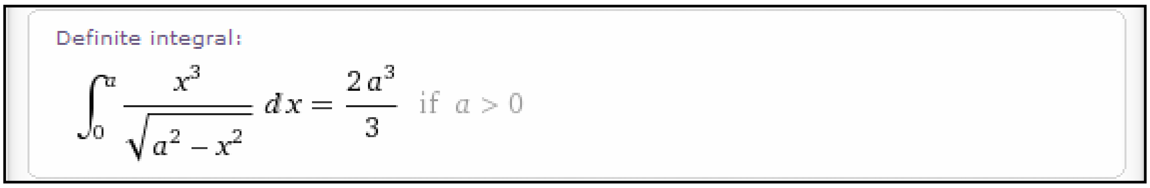
[7.10] Calcular mediante una función de Euler:

$$\int_0^a \frac{x^3}{\sqrt{a^2-x^2}}dx \qquad (a>0)$$

Solución

$$\int_0^a \frac{x^3}{\sqrt{a^2-x^2}}dx = \left[\begin{array}{ll} x=at^{1/2} & x=0\rightarrow t=0 \\ dx=(a/2)t^{-1/2}dt & x=a\rightarrow t=1 \end{array}\right] = \frac{a^3}{2}\int_0^1 \frac{t}{\sqrt{1-t}}dt =$$

$$= \frac{a^3}{2}\int_0^1 t\,(1-t)^{-1/2}\,dt = \frac{a^3}{2}\beta(2,1/2) = \frac{a^3}{2}\frac{\Gamma(2)\cdot\Gamma(1/2)}{\Gamma(5/2)} = \frac{a^3}{2}\frac{\Gamma(1/2)}{\frac{3}{2}\frac{1}{2}\Gamma(1/2)} = \frac{2a^3}{3}$$



[7.11] Calcular la siguiente integral:

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx$$

Solución

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx = \begin{bmatrix} x = a\sqrt{t} & x=0\rightarrow t=0 \\ dx=(a/2)t^{-1/2}dt & x=a\rightarrow t=1 \end{bmatrix} = \frac{a^4}{2}\int_0^1 t^{1/2}(1-t)^{1/2}\,dt =$$

$$= \frac{a^4}{2}\beta\left(\frac{3}{2},\frac{3}{2}\right) = \frac{a^4}{2}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} = \frac{a^4}{2}\frac{\left(\frac{1}{2}\sqrt{\pi}\right)^2}{2} = \frac{a^4\pi}{16}$$

[7.12] Sea la función f definida por: $f(x) = \begin{cases} 1-x & \text{si} \quad x<1 \\ \ln x & \text{si} \quad x\geq 1 \end{cases}$

Hallar el área limitada por la curva de ecuación $y = f(x)$ y las rectas de ecuaciones $y=0,\quad x=0$ y $x=e$

Solución

En primer lugar obtendremos una primitiva de la función $f(x)=\ln x$ utilizando el método de integración por partes.

$$\int \ln x\,dx = \begin{bmatrix} \ln x = u \rightarrow du = \frac{dx}{x} \\ dx = dv \rightarrow v = x \end{bmatrix} = x\ln x - \int x\frac{1}{x}\,dx = x\ln x - x + cte$$

Por lo tanto:

$$A = \int_0^1 (1-x)dx + \int_1^e \ln x\,dx = \left[x-\frac{x^2}{2}\right]_0^1 + \left[x\ln x - x\right]_1^e = 1-\frac{1}{2}+e\ln e - e - \ln 1 + 1 = 2-\frac{1}{2} = \frac{3}{2} \quad u^2$$

Input: *Mathematica form*

$$\int_0^1 (1-x)\, dx + \int_1^e \log(x)\, dx$$

$\log(x)$ is the natural logarithm »

Result: More digits

$$\frac{3}{2} \approx 1.5\ldots$$

[7.13] Hallar el área determinada por la curva de ecuación $y=\dfrac{1}{1+x^2}$ y su asíntota.

Solución

$$A=2\int_0^{\infty}\frac{1}{1+x^2}dx=2\lim_{u\to\infty}\int_0^{u}\frac{1}{1+x^2}dx=2\lim_{u\to\infty}\operatorname{arctg}x\Big]_0^u=$$

$$=2\lim_{u\to\infty}\left(\operatorname{arctg}u-\operatorname{arctg}0\right)=2\left(\frac{\pi}{2}-0\right)=\pi \quad u^2$$

Input: *Mathematica form*

$$2\int_0^{\infty}\frac{1}{1+x^2}\, dx$$

Result: More digits

$$\pi \approx 3.14159\ldots$$

[7.14] Hallar el área de la región limitada entre las curvas de ecuaciones cartesianas $y=\dfrac{1}{x^2-x}$ y $y=\dfrac{1}{x^3}$ en los siguientes casos:

1) entre las abscisas $a=2$ y $b=3$
2) para la región dada por $x\geq 3$

Solución

1) Para $x\in(2,3)$ se cumple que $x^3>x^2-x \;\Rightarrow\; \dfrac{1}{x^3}<\dfrac{1}{x^2-x}$

$$A=\int_2^3\left(\frac{1}{x^2-x}-\frac{1}{x^3}\right)dx=\int_2^3\left(\frac{1}{x(x-1)}-\frac{1}{x^3}\right)dx=2\int_2^3\left(\frac{1}{x-1}-\frac{1}{x}-\frac{1}{x^3}\right)dx=\ln(x-1)-\ln x-\frac{x^{-2}}{-2}\Big]_2^3=$$

$$= \ln(x-1) - \ln x + \frac{1}{2x^2}\Big]_2^3 = \ln 2 - \ln 3 + \frac{1}{18} - \ln 1 + \ln 2 - \frac{1}{8} = 2\ln 2 - \ln 3 - \frac{5}{72} = \ln\frac{4}{3} - \frac{5}{72} \quad u^2$$

Definite integral: More digits

$$\int_2^3 \left(\frac{1}{x^2-x} - \frac{1}{x^3}\right) dx = -\frac{5}{72} + \log\left(\frac{4}{3}\right) \approx 0.218238\ldots$$

$\log(x)$ is the natural logarithm »

2) También para $x \geq 3$ se cumple que $\frac{1}{x^3} < \frac{1}{x^2 - x}$

$$A = \int_3^\infty \left(\frac{1}{x^2-x} - \frac{1}{x^3}\right) dx = \lim_{p\to\infty} \int_3^p \left(\frac{1}{x(x-1)} - \frac{1}{x^3}\right) dx = \lim_{p\to\infty} \left[\ln(x-1) - \ln x + \frac{1}{2x^2}\right]_3^p =$$

$$= \lim_{p\to\infty}\left[\ln(p-1) - \ln p + \frac{1}{2p^2}\right] - \ln 2 + \ln 3 - \frac{1}{18} = \lim_{p\to\infty}\left[\ln\frac{p-1}{p} + \frac{1}{2p^2}\right] + \ln\frac{3}{2} - \frac{1}{18} = \ln\frac{3}{2} - \frac{1}{18}$$

Definite integral: More digits

$$\int_3^\infty \left(\frac{1}{x^2-x} - \frac{1}{x^3}\right) dx = -\frac{1}{18} + \log\left(\frac{3}{2}\right) \approx 0.34991\ldots$$

$\log(x)$ is the natural logarithm »

[7.15] Hallar el valor medio integral μ y el valor medio cuadrático integral μ_c en [-1,1] de la función f definida por: $f(x) = e^x \quad \forall x \in [-1,1]$

Solución

El valor medio integral μ en un intervalo $[a,b]$ es: $\mu = \frac{1}{b-a}\int_a^b f(x)dx$

$$\mu = \frac{1}{2}\int_{-1}^1 e^x dx = \frac{1}{2}\, e^x\Big]_{-1}^1 = \frac{1}{2}\left(e - \frac{1}{e}\right)$$

Input: *Mathematica* form

$$\frac{1}{2}\int_{-1}^1 e^x\, dx$$

Result: More digits

$$\frac{1}{2}\left(-\frac{1}{e} + e\right) \approx 1.1752\ldots$$

El valor medio cuadrático integral μ_c en un intervalo $[a,b]$ es: $\mu_c = \sqrt{\dfrac{1}{b-a}\int_a^b [f(x)]^2\,dx}$

$$\mu_c = \sqrt{\frac{1}{2}\int_{-1}^{1} e^{2x}dx} = \frac{1}{\sqrt{2}}\sqrt{\left.\frac{e^{2x}}{2}\right]_{-1}^{1}} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\sqrt{e^2-e^{-2}} = \frac{1}{2}\sqrt{e^2-\frac{1}{e^2}} = \frac{\sqrt{e^4-1}}{2e}$$

[7.16] Sea la función f definida en [0,2] por:

$$f(x) = \begin{cases} x & \text{si} \quad 0 \le x < 1 \\ 1/x & \text{si} \quad 1 \le x \le 2 \end{cases}$$

1) Hallar la altura de un rectángulo de base 2 cuya área coincida con el área del recinto limitado por la curva de ecuación cartesiana $y = f(x)$, el eje OX y las rectas de ecuaciones $x = 0$ y $x = 2$.
2) Determinar el radio de la base de un cilindro circular de altura 2 cuyo volumen coincida con el volumen engendrado al girar la curva de ecuación cartesiana $y = f(x)$ alrededor del eje OX entre los puntos de abscisa $a = 0$ y $a = 2$.

Solución

1) La altura pedida es el valor medio integral μ

$$\mu = \frac{1}{2}\int_0^1 x\,dx + \frac{1}{2}\int_1^2 \frac{1}{x}dx = \frac{1}{2}\left.\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\ln x\Big]_1^2 = \frac{1}{2}\frac{1}{2} + \frac{1}{2}\ln 2 = \frac{1}{4} + \frac{1}{2}\ln 2$$

Input: *Mathematica form*

$$\frac{1}{2}\int_0^1 x\,dx + \frac{1}{2}\int_1^2 \frac{1}{x}\,dx$$

Result: More digits

$$\frac{1}{4} + \frac{\log(2)}{2} \approx 0.596574\ldots$$

$\log(x)$ is the natural logarithm »

2) El radio del cilindro coincide con el valor medio cuadrático integral μ_c

$$(\mu_c)^2 = \frac{1}{2}\int_0^1 x^2dx + \frac{1}{2}\int_1^2 \frac{1}{x^2}dx = \frac{1}{2}\left.\frac{x^3}{3}\right]_0^1 + \left.\frac{1}{2}\frac{-1}{x}\right]_1^2 = \frac{1}{2}\frac{1}{3} + \frac{1}{2}\left(-\frac{1}{2}+1\right) = \frac{5}{12}$$

Por lo tanto, $\mu_c = \sqrt{\dfrac{5}{12}} = \dfrac{1}{2}\sqrt{\dfrac{5}{3}}$

[7.17] Calcular, mediante un desarrollo en serie, la integral: $\displaystyle\int_0^x \frac{1-\cos x}{x}\,dx$

Solución

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^{n-1}\frac{x^{2n-2}}{(2n-2)!}$$

$$1-\cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + (-1)^{n}\frac{x^{2n-2}}{(2n-2)!}$$

$$\int_0^x \frac{1-\cos x}{x}\,dx = \int_0^x \left[\frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + (-1)^{n}\frac{x^{2n-2}}{(2n-2)!}}{x}\right] dx =$$

$$= \int_0^x \left[\frac{x}{2!} - \frac{x^3}{4!} + \frac{x^5}{6!} + \cdots + (-1)^{n}\frac{x^{2n-3}}{(2n-2)!}\right] dx =$$

$$= \frac{x^2}{2\cdot 2!} - \frac{x^4}{4\cdot 4!} + \frac{x^6}{6\cdot 6!} + \cdots + (-1)^{n}\frac{x^{2n-2}}{(2n-2)\cdot(2n-2)!}$$

Definite integral:

$$\int_0^x \frac{1-\cos(x)}{x}\,dx = -\text{Ci}(x) + \log(x) + \gamma$$

$\text{Ci}(x)$ is the cosine integral »

γ is the Euler–Mascheroni constant »

$\log(x)$ is the natural logarithm »

Series expansion at x=0: More terms

$$\frac{x^2}{4} - \frac{x^4}{96} + \frac{x^6}{4320} - \frac{x^8}{322560} + \frac{x^{10}}{36288000} - \frac{x^{12}}{5748019200} + O\left(x^{13}\right) + \text{constant}$$