2. LESSON: SOLVED EXERCISES

1. Let be X a discrete random variable with $p(x)$ as probability function:

$$p(x)=\begin{cases} 2m & x=1 \\ m & x=2,3,4 \\ 0 & \text{other cases} \end{cases}$$

a) Determine the value of the constant m.

b) Get the characteristic function of the X discrete random variable

c) Determine the mean of the X discrete random variable using the characteristic function.

a)

In order to be a probability function:

1) $p(x)\geq 0 \quad \forall x\in\mathbb{R}\Rightarrow m\geq 0$

2) $\sum_{i=1}^{n} p(x_i)=1\Rightarrow 2m+m+m+m=1;\ 5m=1;\ \boxed{m=\frac{1}{5}}$

b)

$$\Psi(t)=\sum_{k=1}^{n} e^{itx_k}\cdot p(x_k)=e^{it}\cdot\frac{2}{5}+e^{2it}.\frac{1}{5}+e^{3it}.\frac{1}{5}+e^{4it}.\frac{1}{5}=\boxed{\frac{1}{5}\left(2e^{it}+e^{2it}+e^{3it}+e^{4it}\right)}$$

c)

As the mean is the first-order moment:

$$\alpha_1=\frac{1}{i}\frac{d\Psi(t)}{dt}\bigg|_{t=0}=\frac{1}{i}\frac{1}{5}\left(2ie^{it}+2ie^{2it}+3ie^{3it}+4ie^{4it}\right)\bigg|_{t=0}=\frac{1}{5}(2+2+3+4)=\boxed{\frac{11}{5}}$$

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2. The kinetics of a chemical reaction want to be studied and it has been proven that the density function for the reagent consumption (mol/min) continuous random variable is the corresponding:

$$f(x)=\begin{cases} ke^{-x} & x>0 \\ 0 & x\leq 0 \end{cases}$$

a) Which is the value of k constant for $f(x)$ to be a density function?

b) Get the distribution function of the continuous random variable.

c) Determine the probability that the reaction speed (reagent consumption) is greater than 10 mol/min.

a)

In order to be a density function:

1) $f(x)\geq 0 \quad \forall x\in\mathbb{R}\Rightarrow k\geq 0$

2)

$$\int_{-\infty}^{+\infty} f(x)dx=1 \Rightarrow \int_{-\infty}^{0} 0dx+\int_{0}^{+\infty} ke^{-x}dx=1;\ 0+\lim_{t\to\infty}\int_{0}^{t} ke^{-x}=1;\ \lim_{t\to\infty}-ke^{-x}\Big|_{0}^{t}=1;\ \lim_{t\to\infty}\left(-ke^{-t}+ke^{0}\right)=1$$

$$\boxed{k=1}$$

b)

$x\leq 0$:

$$\int_{-\infty}^{x} f(t)dt=\int_{-\infty}^{x} 0dt=0$$

;

$x>0$:

$$\int_{0}^{x} f(t)dt=\int_{-\infty}^{x} e^{-t}dt=-e^{-t}\Big|_{0}^{x}=1-e^{-x}$$

Therefore, distribution function:

$$F(x)=\begin{cases} 1-e^{-x} & x>0 \\ 0 & x\leq 0 \end{cases}$$

c)

$$P(X>10)=1-P(X\leq 10)=1-F(10)=1-(1-e^{-10})=\boxed{e^{-10}}$$

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3. A discrete random variable X can take the values -1, 0 and 1 with the same probability.

   a) Get the moments generating function of the variable X.

   b) Calculate the first four moments with respect to the origin of the variable X.

a)

$$\alpha(w)=E(e^{wx})=e^{w(-1)}\left(\frac{1}{3}\right)+e^{w(0)}\left(\frac{1}{3}\right)+e^{w(1)}\left(\frac{1}{3}\right)=\boxed{\frac{1}{3}\left(1+e^{-w}+e^{w}\right)}$$

b) For obtaining the moments with respect to the origin, the generating function will be derived and evaluated at $w=0$.

First-order moment: $$\left.\frac{dE(e^{wx})}{dw}\right|_{w=0}=\left.\frac{1}{3}\left(0-e^{-w}+e^{w}\right)\right|_{w=0}=0$$

Second-order moment: $$\left.\frac{d^2E(e^{wx})}{dw^2}\right|_{w=0}=\left.\frac{1}{3}\left(0+e^{-w}+e^{w}\right)\right|_{w=0}=\frac{2}{3}$$

Third-order moment: $$\left.\frac{d^3E(e^{wx})}{dw^3}\right|_{w=0}=\left.\frac{1}{3}\left(0-e^{-w}+e^{w}\right)\right|_{w=0}=0$$

Fourth-order moment: $$\left.\frac{d^4E(e^{wx})}{dw^4}\right|_{w=0}=\left.\frac{1}{3}\left(0+e^{-w}+e^{w}\right)\right|_{w=0}=\frac{2}{3}$$

As it can be observed, odd moments are 0 and pair moments are $\frac{2}{3}$.

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4. Let be X a continuous random variable with $f(x)$ as density function.

$$f(x)=\begin{cases}\dfrac{1}{3} & 0\leq x\leq 3\\ 0 & \text{other cases}\end{cases}$$

a) Get the characteristic function of the random variable.

b) Get the moments generating function of the random variable.

c) Calculate the mean of the random variable.

a)

$$\Psi(t)=E\left(e^{itx}\right)=\int_{-\infty}^{+\infty}e^{itx}\,f(x)dx=\int_{-\infty}^{0}e^{itx}0dx+\int_{0}^{3}e^{itx}\frac{1}{3}dx+\int_{3}^{+\infty}e^{itx}0dx=\frac{1}{3}\frac{e^{itx}}{it}\bigg|_{0}^{3}=\frac{1}{3it}\left(e^{3it}-e^{0it}\right)=\boxed{\frac{e^{3it}-1}{3it}}$$

b)

$$\alpha(w)=E\left(e^{wx}\right)=\int_{-\infty}^{+\infty}e^{wx}\,f(x)dx=\int_{-\infty}^{0}e^{wx}0dx+\int_{0}^{3}e^{wx}\frac{1}{3}dx+\int_{3}^{+\infty}e^{wx}0dx=\frac{1}{3}\frac{e^{wx}}{w}\bigg|_{0}^{3}=\frac{1}{3w}\left(e^{3w}-e^{0w}\right)=\boxed{\frac{e^{3w}-1}{3w}}$$

c)

$$\alpha_1=\frac{d\alpha(w)}{dw}\bigg|_{w=0}=\frac{\left(3e^{3w}\cdot 3w\right)-\left(\left(e^{3w}-1\right)\cdot 3\right)}{9w^2}\Bigg|_{w=0}=\frac{3\left(3we^{3w}\right)-\left(\left(e^{3w}-1\right)\right)}{9w^2}\Bigg|_{w=0}=\frac{(3w-1)e^{3w}+1}{3w^2}\bigg|_{w=0}=\frac{0}{0}$$

It results in an indeterminate form, so L'Hôpital's rule must be applied. In this case, the rule has to be applied twice to get the mean.

$$\alpha_1=\frac{(27w+9)e^{3w}}{6}\bigg|_{w=0}=\frac{9}{6}=\boxed{\frac{3}{2}}$$

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5. Let be X a random variable with the following characteristic function:

$$\Psi(t) = k + mt + pt^2$$

a) Get the values of k, m and p, for the mean of the random variable X to be 1 and the variance 4.

b) Get the characteristic function of the random variable $2X$.

c) Get the characteristic function of the random variable $Y = 3X + 2$.

a)

As the zero-order moments have the value of 1:

$$\Psi(0) = k + m0 + p0^2 = k$$

$$\boxed{k = 1}$$

Mean is the first-order moment so:

$$\alpha_1 = \frac{1}{i}\frac{d\Psi(t)}{dt}\bigg|_{t=0} = \frac{1}{i}(m + 2pt)\big|_{t=0} = \frac{m}{i}$$

As the value of the mean is 1:

$$\frac{m}{i} = 1; \quad \boxed{m = i}$$

Variance is the second-order moment centred in the mean, so:

$$\sigma^2 = \alpha_2 - \alpha_1^2 = \frac{1}{i^2}\frac{d^2\Psi(t)}{dt^2}\bigg|_{t=0} - 1^2 = -1(2p)\big|_{t=0} - 1 = -2p - 1$$

As the value of the variance is 4:

$$-2p - 1 = 4; \quad \boxed{p = -\frac{5}{2}}$$

b)

$$\Psi_{2X}(t) = E\left(e^{it2x}\right) = 1 + i(2t) - \frac{5}{2}(2t)^2 = \boxed{1 + 2it - 10t^2}$$

c)

$$\Psi_Y(t) = E\left(e^{ity}\right) = E\left(e^{it(3x+2)}\right) = E\left(e^{it3x}\cdot e^{2it}\right) = e^{2it}\cdot\left(1 + i(3t) - \frac{5}{2}(3t)^2\right) = \boxed{e^{2it}\cdot\left(1 + 3it - \frac{45}{2}t^2\right)}$$

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