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1. LESSON: SOLVED EXERCISES

1. X is a continuous random variable with E(X) mean and $\sigma_x^{\ 2}$ variance. Determine Y random variable's mean and variance knowing that Y=aX+b.

$E(Y)?$

$\sigma_y^{\ 2}?$

$$E(Y)=E(aX+b)=\int_{-\infty}^{\infty}(ax+b)\,\mathrm{f}(x)dx=\int_{-\infty}^{\infty}ax\,\mathrm{f}(x)dx+\int_{-\infty}^{\infty}b\,\mathrm{f}(x)dx=a\int_{-\infty}^{\infty}x\,\mathrm{f}(x)dx+b\int_{-\infty}^{\infty}\mathrm{f}(x)dx=$$

$$=aE(X)+b$$

$$E(Y)=aE(X)+b$$

$$\sigma_y^2=E(Y-E(Y))^2=E((aX+b)-(aE(X)+b))^2$$

$$\sigma_y^2=E(aX+b-aE(X)-b)^2=E(aX-aE(X))^2=aE(X-E(X))^2=a\sigma_x^2$$

$$\sigma_y^2=a\sigma_x^2$$

2. The random variable can take the following discrete values: {2,3,7,8,10}. Knowing that all events have the same probability, calculate first raw moment, second raw moment, first central moment, second central moment, first moment when a=4 and second moment when a=4.

$$\alpha_1 = E(X)^1 = E(X) = \sum_{i=1}^{n} x_i p(x_i) = 2\cdot\frac{1}{5} + 3\cdot\frac{1}{5} + 7\cdot\frac{1}{5} + 8\cdot\frac{1}{5} + 10\cdot\frac{1}{5} = 6 \qquad \alpha_1 = 6$$

$$\alpha_2 = E(X)^2 = \sum_{i=1}^{n} (x_i)^2 p(x_i) = 2^2\cdot\frac{1}{5} + 3^2\cdot\frac{1}{5} + 7^2\cdot\frac{1}{5} + 8^2\cdot\frac{1}{5} + 10^2\cdot\frac{1}{5} = 45.2 \qquad \alpha_2 = 45.2$$

$$\mu_1 = E(X - E(X))^1 = \sum_{i=1}^{n} (x_i - E(X)) p(x_i) =$$

$$\mu_1 = (2-6)\cdot\frac{1}{5} + (3-6)\cdot\frac{1}{5} + (7-6)\cdot\frac{1}{5} + (8-6)\cdot\frac{1}{5} + (10-6)\cdot\frac{1}{5} = 0 \qquad \mu_1 = 0$$

$$\mu_2 = E(X - E(X))^2 = \sum_{i=1}^{n} (x_i - E(X))^2 p(x_i) =$$

$$\mu_2 = (2-6)^2\cdot\frac{1}{5} + (3-6)^2\cdot\frac{1}{5} + (7-6)^2\cdot\frac{1}{5} + (8-6)^2\cdot\frac{1}{5} + (10-6)^2\cdot\frac{1}{5} = 9.2 \qquad \mu_2 = \sigma^2 = 9.2$$

$$\alpha_{1,4} = E(X-4)^1 = \sum_{i=1}^{n} (x_i - 4) p(x_i) =$$

$$\alpha_{1,4} = (2-4)\cdot\frac{1}{5} + (3-4)\cdot\frac{1}{5} + (7-4)\cdot\frac{1}{5} + (8-4)\cdot\frac{1}{5} + (10-4)\cdot\frac{1}{5} = 2 \qquad \alpha_{1,4} = 2$$

$$\alpha_{2,4} = E(X-4)^2 = \sum_{i=1}^{n} (x_i - 4)^2 p(x_i) =$$

$$\alpha_{2,4} = (2-4)^2\cdot\frac{1}{5} + (3-4)^2\cdot\frac{1}{5} + (7-4)^2\cdot\frac{1}{5} + (8-4)^2\cdot\frac{1}{5} + (10-4)^2\cdot\frac{1}{5} = 13,2 \qquad \alpha_{2,4} = 13.2$$

3. The random variable can take the following discrete values: {2,3,7,8,10}. Knowing that all events have the same probability, calculate skewness and kurtosis values, and describe briefly the shape of the distribution.

$$\gamma_1 = \frac{1}{\sigma^3} E(X - E(X))^3 = \frac{\mu_3}{\sigma^3}$$

$$\mu_3 = E(X - E(X))^3 = \sum_{i=1}^{n} (x_i - E(X))^3 p(x_i) =$$

$$\mu_3 = (2-6)^3 \cdot \frac{1}{5} + (3-6)^3 \cdot \frac{1}{5} + (7-6)^3 \cdot \frac{1}{5} + (8-6)^3 \cdot \frac{1}{5} + (10-6)^3 \cdot \frac{1}{5} = -3.6 \qquad \mu_3 = -3.6$$

$$\mu_2 = \sigma^2 = E(X - E(X))^2 = \sum_{i=1}^{n} (x_i - E(X))^2 p(x_i) =$$

$$\sigma^2 = (2-6)^2 \cdot \frac{1}{5} + (3-6)^2 \cdot \frac{1}{5} + (7-6)^2 \cdot \frac{1}{5} + (8-6)^2 \cdot \frac{1}{5} + (10-6)^2 \cdot \frac{1}{5} = 9.2 \qquad \sigma^2 = 9.2$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{9.2}$$

$$\gamma_1 = \frac{\mu_3}{\sigma^3} = \frac{-3.6}{\left(\sqrt{9.2}\right)^3} = -0.13$$

The value is very near to 0, so is slightly negative skew

$$\gamma_2 = \frac{\mu_4}{\sigma^4} - 3$$

$$\mu_4 = E(X - E(X))^4 = \sum_{i=1}^{n} (x_i - E(X))^4 p(x_i) =$$

$$\mu_4 = (2-6)^4 \cdot \frac{1}{5} + (3-6)^4 \cdot \frac{1}{5} + (7-6)^4 \cdot \frac{1}{5} + (8-6)^4 \cdot \frac{1}{5} + (10-6)^4 \cdot \frac{1}{5} = 122 \qquad \mu_4 = 122$$

$$\mu_2 = \sigma^2 = E(X - E(X))^2 = \sum_{i=1}^{n} (x_i - E(X))^2 p(x_i) =$$

$$\sigma^2 = (2-6)^2 \cdot \frac{1}{5} + (3-6)^2 \cdot \frac{1}{5} + (7-6)^2 \cdot \frac{1}{5} + (8-6)^2 \cdot \frac{1}{5} + (10-6)^2 \cdot \frac{1}{5} = 9.2 \qquad \sigma^2 = 9.2$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{9.2}$$

$$\gamma_2 = \frac{\mu_4}{\sigma^4} - 3 = \frac{122}{\left(\sqrt{9.2}\right)^4} - 3 = -1.56$$

The value is negative so it could be said that the distribution is slightly platykurtic.

4. The mean of a random variable is 7 and second central moment is 4. Calculate the minimum probability that the random variable is within (4, 14).

$$P(E(X)-k\sigma < X < E(X)+k\sigma) \geq 1-\frac{1}{k^2} \quad \forall k>0$$

$$P(4 < X < 10) \geq 1-\frac{1}{k^2} \quad \forall k>0$$

$$4 = E(X)-k\sigma \qquad 4=7-k2 \qquad k=\frac{3}{2} \quad \text{or}$$

$$10 = E(X)+k\sigma \qquad 10=7+k2 \qquad k=\frac{3}{2}$$

$$P(4 < X < 10) \geq 1-\frac{1}{\left(\frac{3}{2}\right)^2} = \frac{5}{9}$$

5. The pedestrian waiting time in a traffic light is a continuous random variable. A person arrives randomly at the crossroad, what is the average waiting time? The density function of the random variable is (values are in seconds) is the following:

$$\mathrm{f}(x) = \begin{cases} 0 & -\infty < x < 0 \\ \dfrac{x}{80} & 0 < x < 30 \\ 0 & 80 < x < \infty \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x\,\mathrm{f}(x)dx = \int_{0}^{30} x\frac{x}{80}dx = \int_{0}^{30} \frac{x^2}{80}dx = \left.\frac{x^3}{240}\right]_0^{30} = \left.\frac{30^3}{240}\right] = 112.5 \text{ seconds}$$