FUNCTIONS OF ONE-DIMENSIONAL RANDOM VARIABLE

2. LESSON

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OBJECTIVES

- ✓ Define the functions of probability, density and distribution of a random variable and identify the differences between them
- ✓ Understand the concept of characteristic function of a random variable and be able to calculate different moments through it
- ✓ Understand the concept of generating function of a random variable and be able to calculate different moments through it

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2.1. Probability function

I. Definition

- Let be X a discrete random variable. The probability (or mass) function, $p(x)$, is defined as:

$$p:\ \mathbb{R} \to [0,1]$$
$$x \to p(x) = P(X = x)$$

- In this way the sample space (S) of a discrete random variable is defined as:

$$S_x = \{x \in \mathbb{R} : p(x) > 0\}$$

- Probability function, $p(x)$, assigns to each point of the sample space, S_x, its probability and will be 0 for any external point of the sample space.

II. Properties

1. $p(x) \geq 0, \quad \forall x \in \mathbb{R}$

2. $\sum_{i=1}^{n} p(x_i) = 1$

3. Let be $A \subset \mathbb{R}$, so $P(A) = \sum_{x_i \in A} p(x_i)$

2.2. Density function

I. Definition

- Let be X a continuous random variable. The density function, $\mathrm{f}(x)$, of the X random variable is defined as:

$$\mathrm{f}: \mathbb{R} \to \mathbb{R}$$

$$P(a < x < b) = \int_a^b \mathrm{f}(x)dx \qquad a, b \in \mathbb{R} \text{ being } a < b.$$

- In this way the sample space (S) of a continuous random variable is defined as:

$$S_x = \{x \in \mathbb{R} : \mathrm{f}(x) > 0\}$$

- The probability that a continuous random variable X takes a given value is 0 meaning, $P(X = x) = 0 \quad \forall x \in \mathbb{R}$.

II. Properties

1. $f(x) \geq 0, \quad \forall x \in \mathbb{R}$

2. $\int_{-\infty}^{+\infty} f(x)dx = 1$

3. Let be $A \subset \mathbb{R}$, so $P(A) = \int_A f(x)dx$

2.3. Distribution function

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I. Definition

- Let be X a random variable. The distribution function, $F(x)$, of that random variable is defined as:

$$F:\ \mathbb{R} \to [0,1]$$
$$x \to F(x) = P(X \le x)$$

- Although the definition of the distribution function is the same, its calculation depends on the nature (discrete or continuous) of the random variable.

- **In discrete case**: $F(x) = P(X \le x) = \sum_{x_i \le x} p(x_i) \quad \forall x \in \mathbb{R}$

I. Definition

- In discrete cases, the probability function can be obtained from the distribution function as follows :

$$p(x_i) = F(x_i) - F(x_{i-1})$$

- **In continuous case:** $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt \quad \forall x \in \mathbb{R}$

- In continuous cases, the probability function can be obtained from the distribution function as follows:

$$f(x) = \frac{dF(x)}{dx}$$

II. Properties

1. $\lim\limits_{x\to-\infty} F(x)=0$

2. $\lim\limits_{x\to\infty} F(x)=1$

3. The distribution function is non-descending:

$$F(x_1)\leq F(x_2) \quad \forall x_1, x_2 \in \mathbb{R} \;\; x_1 < x_2$$

4. The distribution function is right continuous.

2.4. Characteristic function

I. Definition

- Let be X a random variable and "t" a real parameter. The characteristic function of the X random variable is designated with the symbol $\Psi(t)$. It is a parametric function that by definition has the following aspect:

$$\Psi(t) = E\left(e^{itx}\right)$$

Where "e" is Euler's number and "i" is the imaginary unit.

- Consequently, using the Euler formula, the trigonometric expression of this complex number is obtained as:

$$e^{itx} = \cos(tx) + i\sin(tx)$$

I. Definition

- If the random variable is continuous, being $f(x)$ its density function:

$$\Psi(t) = \int_{-\infty}^{+\infty} e^{itx} \cdot f(x)\, dx$$

- If the random variable is discrete, being $p(x)$ its probability function:

$$\Psi(t) = \sum_{j=1}^{n} e^{itx_j} \cdot p(x_j)$$

- As can be seen, the characteristic function in the case of continuous random variable, is the Fourier transform of the density function.

- Similarly, in the case of the discrete random variable, the characteristic function can be obtained by Fourier's serial development of the probability function.

I. Definition

Theorem of uniqueness:

Let be $F(x), G(x)$ distribution functions, which have $f(t), g(t)$ as characteristic functions, respectively. Let's assume that $f(t) = g(t) \quad \forall t \in \mathbb{R}$. Then, for every value of x, $F(x) = G(x)$.

- As a result of this theorem, the characteristic function of a random variable uniquely defines its distribution function. That is, all variables that follow the same distribution function have the same characteristic function and vice versa.

II. Properties

1. When the parameter has a null value, namely, when, $t=0$, the characteristic function has a value of 1.

$$\Psi(0)=E\left(e^{i0x}\right)=E\left(e^{0}\right)=E\left(1\right)=1$$

2. Since the characteristic function is parametric, it is continuous for every t.

3. Calculation of the characteristic function of a linear transformation:

Let be $Y=aX+b$

where X is a random variable with $\Psi_X(t)$ as characteristic function.

$$\Psi_Y(t)=E\left(e^{ity}\right)=E\left(e^{it(ax+b)}\right)=E\left(e^{itax}\cdot e^{itb}\right)$$

II. Properties

Since e^{itb} is a constant, using the properties of the mean:

$$E\left(e^{itax}\cdot e^{itb}\right)=e^{itb}E\left(e^{itax}\right)$$

When the characteristic function of the random variable X takes (ta) value, $E\left(e^{itax}\right)=\Psi_X(ta)$, so:

$$\boxed{\Psi_Y(t)=e^{itb}\cdot\Psi_X(ta)}$$

4. Calculation of the characteristic function of a linear transformation between independent variables:

Let be $Z=aX+bY+c$

II. Properties

The characteristic function of the Z random variable:

$$\Psi_Z(t) = E\left(e^{itz}\right) = E\left(e^{it(ax+by+c)}\right) = E\left(e^{itax} \cdot e^{itby} \cdot e^{itc}\right)$$

As X and Y are independent,

$$\Psi_Z(t) = E\left(e^{itax}\right) \cdot E\left(e^{itby}\right) \cdot E\left(e^{itc}\right)$$

Since e^{itc} is a constant, $E\left(e^{itc}\right) = e^{itc}$, $E\left(e^{itax}\right) = \Psi_X(ta)$, namely, when the parameter of the characteristic function of the variable X has (ta) value.

In the same way, $E\left(e^{itby}\right) = \Psi_Y(tb)$, namely, when the parameter of the characteristic function of the variable Y has (tb) value, so:

II. Properties

$$\Psi_Z(t) = e^{itc} \cdot \Psi_X(ta) \cdot \Psi_Y(tb)$$

- This response can be generalised to any number of sums of independent random variables, $X_1, X_2, ..., X_n$:

$$\Psi_X(t) = \Psi_{X_1}(a_1 t) \cdot \Psi_{X_2}(a_2 t) \cdot ... \cdot \Psi_{X_n}(a_n t) = \prod_{j=1}^{n} \Psi_{X_i}(a_j t)$$

5. Calculation of the moments of a random variable by means of derivatives of the characteristic function:

   - First derivative:

$$\Psi'(t) = \frac{dE(e^{itx})}{dt} = E\frac{de^{itx}}{dt} = E(e^{itx})ix$$

II. Properties

- Second derivative:

$$\Psi''(t) = \frac{dE(e^{itx})ix}{dt} = E\,\frac{de^{itx}ix}{dt} = E(e^{itx})(ix)^2$$

- In general, *k*th derivative:

$$\boxed{\Psi^{k)}(t) = \frac{dE(e^{itx})\left(ix\right)^{k-1}}{dt} = E\,\frac{d\left(e^{itx}\right)\left(ix\right)^{k-1}}{dt} = E(e^{itx})(ix)^k}$$

- If the first derivative is evaluated at $t = 0$:

$$\left.\frac{d\Psi(t)}{dt}\right|_{t=0} = E(e^{itx})ix\Big|_{t=0} = iE(x)$$

II. Properties:

- In general:

$$\left.\frac{d\Psi^k(t)}{dt^k}\right|_{t=0} = i^k E(x^k)$$

- So,

$$\boxed{\alpha_k = \frac{1}{i^k}\left.\frac{d\Psi^k(t)}{dt^k}\right|_{t=0} = E(x^k)}$$

- The characteristic function is very useful to calculate moments of any order.

2.5. Generating function

I. Definition

- The generating function of moments is defined as follows:

$$\alpha(w) = E\left(e^{wx}\right)$$

- It has the same appearance as the characteristic function, but unlike the previous one, the generating function may not exist.

- For the generating function to exist $w \in (-a, a) \quad \forall a > 0$. In this way the generating function is derivable at $w = 0$.

- Substituting the parameter w by it the characteristic function is obtained .

$$\Psi(t) = \alpha(iw) = E\left(e^{iwx}\right)$$

I. Definition

- If the random variable is continuous, being $f(x)$ its density function :

$$\alpha(w) = \int_{-\infty}^{+\infty} e^{wx} \cdot f(x)\, dx$$

- If the random variable is discrete, being $p(x)$ its probability function:

$$\alpha(w) = \sum_{j=1}^{n} e^{wx_j} \cdot p(x_j)$$

- In the generating function the theorem of uniqueness is also fulfilled. All variables that follow the same distribution function have the same generating function and vice versa.

II. Properties

1. When the parameter has a null value, namely, when, $w=0$, the generating function has a value of 1.

$$\alpha(0)=E\left(e^{0x}\right)=E\left(e^{0}\right)=E\left(1\right)=1$$

2. As the generating function is parametric, it is continuous for every w.

3. Calculation of the generating function of a linear transformation:

   Let be $Y=aX+b$

   where X is a random variable with $\alpha_X(w)$ as generating function.

$$\alpha_Y(w)=E\left(e^{wy}\right)=E\left(e^{w(ax+b)}\right)=E\left(e^{wax}\cdot e^{wb}\right)$$

II. Properties

Since e^{wb} is a constant, using the properties of the mean:

$$E\left(e^{wax}\cdot e^{wb}\right)=e^{wb}E\left(e^{wax}\right)$$

When the generating function of the random variable X takes (wa) value , $E\left(e^{wax}\right)=\alpha_X(wa)$, so:

$$\boxed{\alpha_Y(a)=e^{wb}\cdot\alpha_X(wa)}$$

4. Calculation of the generating function of a linear transformation between independent variables:

Let be $Z=aX+bY+c$

II. Properties

The generating function of the Z random variable:

$$\alpha_Z(w) = E\left(e^{wz}\right) = E\left(e^{w(ax+by+c)}\right) = E\left(e^{wax} \cdot e^{wby} \cdot e^{wc}\right)$$

As X and Y are independent,

$$\alpha_Z(w) = E\left(e^{wax}\right) \cdot E\left(e^{wby}\right) \cdot E\left(e^{wc}\right)$$

Since e^{wc} is a constant, $E\left(e^{wc}\right) = e^{wc}$, $E\left(e^{wax}\right) = \alpha_X(wa)$, namely, when the parameter of the generating function of the variable X has (wa) value.

In the same way, $E\left(e^{wby}\right) = \alpha_Y(wb)$, namely, when the parameter of the generating function of the variable Y has (wb) value, so:

II. Properties

$$\boxed{\alpha_Z(w) = e^{wc} \cdot \alpha_X(wa) \cdot \alpha_Y(wb)}$$

- This response can be generalised to any number of sums of independent random variables $X_1, X_2, \ldots, X_n$:

$$\alpha(w) = \alpha_{x_1}(a_1 w) + \alpha_{x_2}(a_2 w) + \ldots + \alpha_{x_n}(a_n w) = \prod_{i=1}^{n} \alpha_{x_i}(a_i w)$$

5. Calculation of the moments of a random variable by means of derivatives of the generating function:

  - First derivative:

$$\alpha(t) = \frac{dE(e^{wx})}{dw} = E\frac{de^{wx}}{dw} = E(e^{wx})x$$

II. Properties

- Second derivative:

$$\alpha''(w) = \frac{dE(e^{wx})x}{dw} = E\frac{de^{wx}x}{dw} = E(e^{wx})x^2$$

- In general, *k*th derivative :

$$\boxed{\alpha^{k)}(w) = \frac{dE(e^{wx})x^{k-1}}{dw} = E\frac{d(e^{wx})x^{k-1}}{dw} = E(e^{wx})x^k}$$

- If the first derivative is evaluated at $w = 0$, first-order moment is obtained:

$$\left.\frac{d\alpha(w)}{dw}\right|_{w=0} = E(e^{wx})x\Big|_{w=0} = E(x) \quad \text{First -order moment}$$

II. Properties

- In general, to obtain the moment of order *k*:

$$\left.\frac{d\alpha^k(w)}{dt^k}\right|_{w=0} = E(e^{wx})x^k\Big|_{w=0} = E(x^k)$$