MATHS BASIC COURSE FOR UNDERGRADUATES

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SOLUTIONS: 2nd. SUBJECT. COMPLEX NUMBERS

SOLUTION EXERCISE 1: \(5i = 5\pi/2; 1+i = (\sqrt{2})\pi/4\) and \(-1-i = (\sqrt{2})5\pi/4\).

SOLUTION EXERCISE 2: \((3\pi/3)^3 = 27\pi;\)
\[
\frac{2\pi}{\sqrt{5}\pi/4} = \frac{2}{\sqrt{5}} \pi/12.
\]

SOLUTION EXERCISE 3: Observe that \(z = -\sqrt{3} + i\) can also be written as \(z = 2\cos(5\pi/6) + i\sin(5\pi/6)\). Hence,
\[
z^7 = 2^7(\cos(35\pi/6) + i\sin(35\pi/6)) = 2^7(\cos(-\pi/2) + i\sin(-\pi/2)) = 2^6(\sqrt{3} - i).
\]

SOLUTION EXERCISE 4: First of all, observe that the equation \(\cos 3\theta + i\sin 3\theta = (\cos \theta + i\sin \theta)^3\) is fulfilled. Raising to the third power and matching the real and the imaginary parts of both sides, we conclude that \(\cos 3\theta = 4\cos^3\theta - 3\cos \theta\).

SOLUTION EXERCISE 5: The 4-th roots of the unity are \(1, e^{i\pi/4}, e^{i\pi},\) and \(e^{i\pi/4},\) which can also be written as \(1, i, -1, -i.\) Moreover, the 6-th roots of the unity are \(1, e^{i\pi/3}, e^{i2\pi/3}, -1, e^{i4\pi/3},\) etc \(e^{i5\pi/3},\) and they are the vertices of a regular hexagon inscribed in a circle.

SOLUTION EXERCISE 6: We have to calculate the fifth roots of the complex number \(-\sqrt{3} + i\). To accomplish this, first we have to express \(z_0 = -\sqrt{3} + i = 2e^{i5\pi/6}\). Then, a fifth root of \(z_0\) can be \(\alpha = 2e^{i\pi/5}\). If \(w\) is a fifth root of the unity, then \((\alpha \cdot w)^5 = \alpha^5 w^5 = \alpha^5 = z_0\). Thus, \(\alpha \cdot w\) is also a fifth root of \(z_0\). Consequently, the fifth roots of \(-\sqrt{3} + i\) are
\[
\alpha, \alpha e^{i2\pi/5}, \alpha e^{i4\pi/5}, \alpha e^{i6\pi/5}, \alpha e^{i8\pi/5}.
\]

In fact, there are five roots that are the fifth roots of \(z_0\). If \(\beta\), is another fifth root of \(z_0\), then \(\beta^5 = \alpha^5 = z_0\), and from here, \(\frac{\beta}{\alpha} = 1\), which indicates that \(\beta = \alpha \cdot w\) is a fifth root of the unity. As a consequence, \(\beta = \alpha \cdot w\), which is already on the previous list.

Therefore, all the fifth roots of \((-\sqrt{3} + i)\) are
\[
2^\frac{1}{5}e^{i\pi/5}, 2^\frac{1}{5}e^{i12\pi/5}, 2^\frac{1}{5}e^{i22\pi/5}, 2^\frac{1}{5}e^{i34\pi/5}, 2^\frac{1}{5}e^{i52\pi/5}.
\]