Solution to Exercise 4: R_0 formulation for SIPF and SIPDF models

Method of calculation

As in Lesson 5, to formulate R_0 the NGM method was applied as follows:

1. Find the steady state for the population with no infected individuals or no infectious elements.
2. Linearize the equations around a population with some number of susceptibles ($S = N$), where N is the initial number of susceptibles in the population, and small numbers of infected ($I \ll N$) or infectious elements ($D, P, F \ll N$).
3. Consider the subset of the equations that involve the infectious processes. That is, ignore the $\frac{dS}{dt}$ equation.
4. Separate the terms in the remaining equations into two parts: transmission, represented by a matrix $\mathbf{T}$, and transistions, represented by a matrix .
5. Construct the next-generation matrix (NGM) for the large domain ($\mathbf{K_L} = -\mathbf{T} \cdot \Sigma^{-1}$).
6. Calculate the eigenvalues for $\mathbf{K_L}$. The dominant eigenvalue is Ro.

R_0 for the SIPF model

This and the SIPDF model have been described in detail in the 'Solution to Exercise 3' document.

$$\begin{aligned}\frac{d\,S}{dt} &= -\beta\,F\\ \frac{d\,I}{dt} &= \beta\,F - m\,I\\ \frac{d\,P}{dt} &= c\,I - (r + f\,(S+I))\,P\\ \frac{d\,F}{dt} &= f\,P\,S - a\,F\end{aligned}$$

The equation for a non-infected population ($I = P = F = 0$) is $\frac{d\,S}{dt} = 0$. The population is constant at the initial population level of N.

The linearized equations around the steady state have $S = N$ and $I, P, F \ll N$. The resulting linearized equations for the infection processes are

$$\begin{aligned}\frac{d\,P}{dt} &= c\,I - (r + f\,N)\,P\\ \frac{d\,F}{dt} &= f\,P\,N - a\,F\\ \frac{d\,I}{dt} &= \beta\,F - m\,I\end{aligned}$$

The transmission vector is $\vec{x} = (P, F, I)$. Transmission and transition matrices along with the inverse of the transition matrix are

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & c\\ f\,N & 0 & 0\\ 0 & \beta & 0\end{pmatrix} \quad \Sigma = \begin{pmatrix} -r - f\,N & 0 & 0\\ 0 & -a & 0\\ 0 & 0 & -m\end{pmatrix} \text{ and } \Sigma^{-1} = \begin{pmatrix} \frac{-1}{r+f\,N} & 0 & 0\\ 0 & \frac{-1}{a} & 0\\ 0 & 0 & \frac{-1}{m}\end{pmatrix}$$

The large domain NGM is

$$\mathbf{K_L} = -\mathbf{T}\cdot\Sigma^{-1} = \begin{pmatrix} 0 & 0 & c/m\\ \frac{f\,N}{r+f\,N} & 0 & 0\\ 0 & \beta/a & 0\end{pmatrix}$$

The eigenvalues are the solution of the determinant equation similar to above (basically replacing the diagonal zeros with $= \lambda$ and setting the determinant to zero).

The determinant is calculated by reducing by minors as

$$\begin{vmatrix} -\lambda & 0 & c/m \\ \frac{f\,N}{r+f\,N} & -\lambda & 0 \\ 0 & \beta/a & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 \\ \beta/a & -\lambda \end{vmatrix} - \frac{c}{m} \begin{vmatrix} \frac{f\,N}{r+f\,N} & -\lambda \\ 0 & \beta/a \end{vmatrix} = -\lambda^3 + \frac{c}{a}\frac{\beta}{m}\frac{f\,N}{r+f\,N} = 0$$

The solution is

$$\lambda = \sqrt[3]{\frac{c}{a}\frac{\beta}{m}\frac{f\,N}{r+f\,N}} = Ro.$$

Changing the expression slightly shows again that Ro is the geometric mean of the creation and removal rates:

$$Ro = \sqrt[3]{\frac{\beta\,c}{m\,a}\frac{f\,N}{r+f\,N}}$$

R_0 for the SIPDF model

As seen in the document of the solution to Exercise 3, the ODEs for this model is:

$$\begin{aligned} \frac{d\,S}{dt} &= -\beta\,F \\ \frac{d\,I}{dt} &= \beta\,F - m\,I \\ \frac{d\,D}{dt} &= m\,I - d\,D \\ \frac{d\,P}{dt} &= c\,D - (r + f\,(S+I))\,P \\ \frac{d\,F}{dt} &= f\,P\,S - a\,F \end{aligned}$$

The equation for a non-infected population ($I = D = P = F = 0$) is $\frac{d\,S}{dt} = 0$. The population is constant at the initial population level of N.

The linearized equations around the steady state have $S = N$ and $I, D, P, F \ll N$.

The resulting linearized equations for the infection processes are

$$
\begin{aligned}
\frac{d\,D}{dt} &= m\,I - d\,D \\
\frac{d\,P}{dt} &= c\,D - (r + f\,N)\,P \\
\frac{d\,F}{dt} &= f\,P\,N - a\,F \\
\frac{d\,I}{dt} &= \beta\,F - m\,I
\end{aligned}
$$

The transmission vector is $\vec{x} = (P, F, I, D)$. Transmission and transition matrices along with the inverse of the transition matrix are

$$
\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & c \\ f\,N & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & m & 0 \end{pmatrix} \quad \Sigma = \begin{pmatrix} -r - f\,N & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & -m & 0 \\ 0 & 0 & 0 & -d \end{pmatrix}
$$

$$
\text{and } \Sigma^{-1} = \begin{pmatrix} \frac{-1}{r+f\,N} & 0 & 0 & 0 \\ 0 & \frac{-1}{a} & 0 & 0 \\ 0 & 0 & \frac{-1}{m} & 0 \\ 0 & 0 & 0 & \frac{-1}{d} \end{pmatrix}
$$

The large domain NGM is

$$
\mathbf{K_L} = -\mathbf{T} \cdot \Sigma^{-1} = \begin{pmatrix} 0 & 0 & 0 & c/d \\ \frac{f\,N}{r+f\,N} & 0 & 0 & 0 \\ 0 & \beta/a & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

The eigenvalues are the solution of the determinant equation similar to above (basically replacing the diagonal zeros with $= \lambda$ and setting the determinant to zero).

The determinant is calculated by reducing by minors as

$$\begin{vmatrix} -\lambda & 0 & 0 & c/d \\ \frac{fN}{r+fN} & -\lambda & 0 & 0 \\ 0 & \beta/a & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ \beta/a & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} - \frac{c}{m} \begin{vmatrix} \frac{fN}{r+fN} & -\lambda & 0 \\ 0 & \beta/a & -\lambda \\ 0 & 0 & 1 \end{vmatrix} = -\lambda^4 + \frac{c}{a}\frac{\beta}{d}\frac{fN}{r+fN} = 0$$

The solution is

$$\lambda = \sqrt[4]{\frac{c}{a}\frac{\beta}{d}\frac{fN}{r+fN}} = Ro.$$

Changing the expression slightly shows again that Ro is the geometric mean of the creation and removal rates:

$$Ro = \sqrt[4]{\frac{\beta}{d}\frac{c}{a}\frac{fN}{r+fN}}$$