Universidad del País Vasco
Euskal Herriko Unibertsitatea

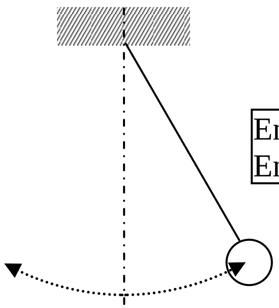
# 8. GAIA.

# BIBRAZIOEN TEORIA

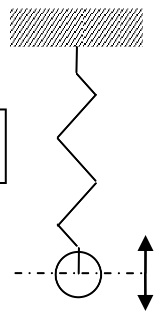
Neftalí Carbajal de la Red

# AURKIBIDEA

## OSZILAZIOA ETA BIBRAZIOA

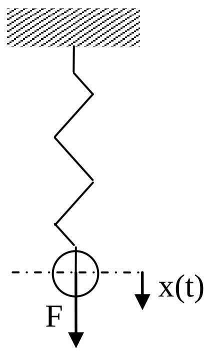


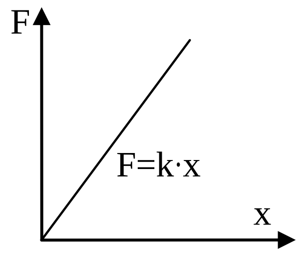
**OSZILAZIOA**
Energia zinetikoa
Energia potentziala



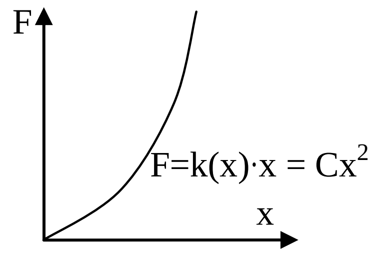
**BIBRAZIOA**
Energia zinetikoa
Energia potentziala
Deformazio energia

## SISTEMA LINEALA – EZ LINEALA



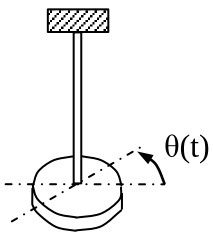

LINEALA



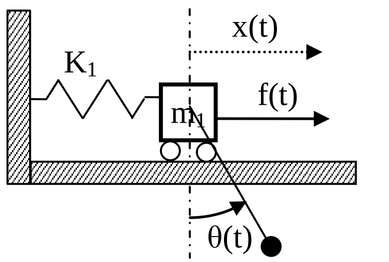

EZ LINEALA

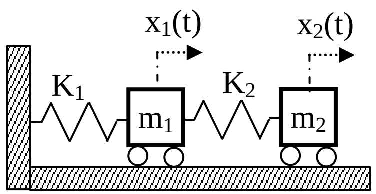
## ASKATASUN MAILA KOPURUA

ASKATASUN MAILA BAKARRA



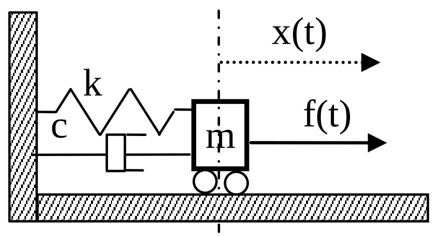

ASKATASUN MAILA BI



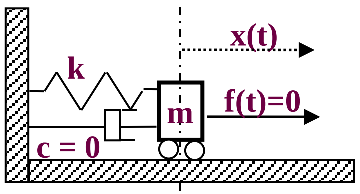

## ASKATASUN MAILA BAKARREKO SISTEMA

$$\sum F_x = 0 \longrightarrow \boxed{m\ddot{x} + c\dot{x} + kx = f(t)}$$




$$\begin{cases} f(t) = 0 & \text{ASKEAK} \\ f(t) \neq 0 & \text{BEHARTUAK} \end{cases}$$

## 2.1. MOTELDU GABEKO BIBRAZIO ASKEAK




HIGIDURAKO EKUAZIOA $\boxed{m\ddot{x}+kx=0}$

EMAITZA $\boxed{x=Ce^{st}}$

EKUAZIO KARAKTERISTIKOA $\boxed{ms^2+k=0}$

HASIERAKO BALDINTZAK $\boxed{x(0)=x_0 \quad \dot{x}(0)=\dot{x}_0}$

MAIZTASUN NATURALA $\boxed{\omega=\sqrt{\dfrac{k}{m}}}$

$$\boxed{s=\pm i\omega}$$

(1) $\boxed{x(t)=C_1e^{i\omega t}+C_2e^{-i\omega t}}$ $\xrightarrow[B=i(C_1-C_2)]{A=C_1+C_2}$ (2) $\boxed{x(t)=A\cos\omega t+B\sin\omega t}$ $\xrightarrow[B=X\sin\theta]{A=X\cos\theta}$ (3) $\boxed{x(t)=X\cos(\omega t-\theta)}$

$$\boxed{\begin{matrix} x(0)=x_0 \\ \dot{x}(0)=\dot{x}_0 \end{matrix}} \longrightarrow \left\{\begin{matrix} A=x_0 \\ B=\dfrac{\dot{x}_0}{\omega} \end{matrix}\right\}$$

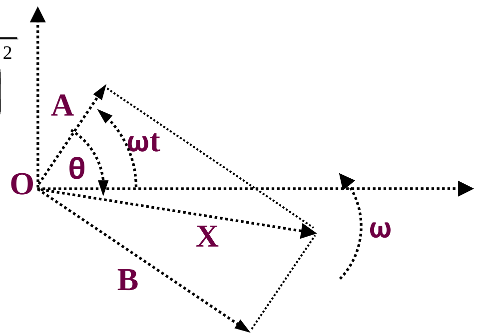
$$X=\sqrt{A^2+B^2} \qquad \theta=arctg\left(\frac{B}{A}\right)$$

$$X=\sqrt{x_0^2+\left(\frac{\dot{x}_0}{\omega}\right)^2} \qquad \theta=arctg\left(\frac{\dot{x}_0}{\omega x_0}\right)$$

$$X=\sqrt{x_0^2+\left(\frac{\dot{x}_0}{\omega}\right)^2}$$

$$A=x_0$$

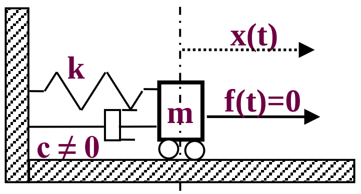
$$B=\frac{\dot{x}_0}{\omega}$$




$$\boxed{x(t)=x_0\cos\omega t+\frac{\dot{x}_0}{\omega}\sin\omega t}$$

$$\boxed{x(t)=\sqrt{x_0^2+\left(\frac{\dot{x}_0}{\omega}\right)^2}\cos\left[\omega t-arctg\left(\frac{\dot{x}_0}{\omega x_0}\right)\right]}$$

## 2.2. MOTELDUTAKO BIBRAZIO ASKEAK




HIGIDURAKO EKUAZIOA $\boxed{m\ddot{x}+c\dot{x}+kx=0}$

EMAITZA $\boxed{x=Ce^{st}}$

EKUAZIO KARAKTERISTIKOA $\boxed{ms^2+cs+k=0}$

HASIERAKO BALDINTZAK $\boxed{\begin{array}{l} x(0)=x_0 \\ \dot{x}(0)=\dot{x}_0 \end{array}}$

MOTELGARRITASUN ERLATIBOA

$$\xi=\frac{c}{\bar{c}}=\frac{c}{2m\omega}$$

$$s_{1,2}=\frac{-c}{2m}\pm\sqrt{\left(\frac{c}{2m}\right)^2-\left(\frac{k}{m}\right)}$$

$\boxed{c>\bar{c}}$ MOTELGARRITASUN GAINKRITIKOA

$$x(t)=e^{-\xi\omega t}\left[C_1e^{\omega t\sqrt{\xi^2-1}}+C_2e^{-\omega t\sqrt{\xi^2-1}}\right]$$

$$\begin{cases} C_1=x_0\left(\dfrac{\xi}{2\sqrt{\xi^2-1}}+\dfrac{1}{2}\right)+\dot{x}_0\left(\dfrac{1}{2\omega\sqrt{\xi^2-1}}\right) \\ C_2=x_0\left(\dfrac{-\xi}{2\sqrt{\xi^2-1}}+\dfrac{1}{2}\right)+\dot{x}_0\left(\dfrac{-1}{2\omega\sqrt{\xi^2-1}}\right) \end{cases}$$

$\boxed{c=\bar{c}=2m\omega}$ MOTELGARRITASUN KRITIKOA

$$x(t)=e^{-\omega t}\left[x_0+t\left(\dot{x}_0+x_0\omega\right)\right]$$

$\boxed{c<\bar{c}}$ MOTELGARRITASUN AZPIKRITIKOA

$$x(t)=e^{-\xi\omega t}\left[x_0\cos\left(\omega_D t\right)+\left(\frac{\dot{x}_0+\xi\omega x_0}{\omega_D}\right)\sin\left(\omega_D t\right)\right]$$

MAIZTASUN MOTELDUA $\omega_D=\omega\sqrt{1-\xi^2}$

$$X=\sqrt{x_0^2+\left(\frac{\dot{x}_0+\xi\omega x_0}{\omega_D}\right)^2}\qquad\theta=arctg\left(\frac{\dot{x}_0+\xi\omega x_0}{x_0\omega_D}\right)$$

### IRUDIKAPEN GRAFIKOA

$\boxed{\xi = 0}$

$$\begin{cases} x(t) = x_0 \cos \omega t + \dfrac{\dot{x}_0}{\omega} \sin\omega t \\ x(t) = \sqrt{x_0^2 + \left(\dfrac{\dot{x}_0}{\omega}\right)^2} \cos\left[\omega t - arctg\left(\dfrac{\dot{x}_0}{\omega x_0}\right)\right] \end{cases}$$

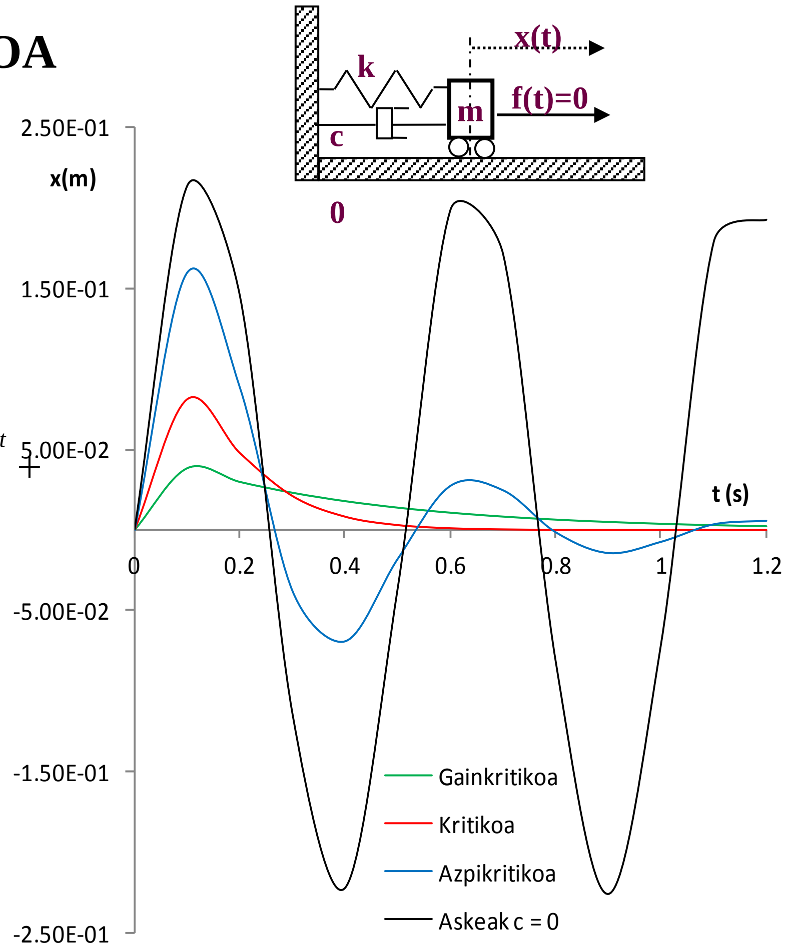
MOTELGARRITASUN GAINKRITIKOA $\boxed{\xi = 2{,}5}$

$$\begin{cases} x(t) = \left[x_0\left(\dfrac{\xi}{2\sqrt{\xi^2-1}} + \dfrac{1}{2}\right) + \dot{x}_0\left(\dfrac{1}{2\omega\sqrt{\xi^2-1}}\right)\right] e^{\left[-\xi\omega+\omega\sqrt{\xi^2-1}\right]t} + \\ + \left[x_0\left(\dfrac{-\xi}{2\sqrt{\xi^2-1}} + \dfrac{1}{2}\right) + \dot{x}_0\left(\dfrac{-1}{2\omega\sqrt{\xi^2-1}}\right)\right] e^{\left[-\xi\omega-\omega\sqrt{\xi^2-1}\right]t} \end{cases}$$

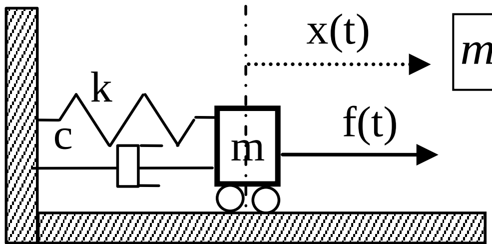
MOTELGARRITASUN KRITIKOA $\boxed{\xi = 1}$ $\longrightarrow$ $x(t) = e^{-\omega t}\left[x_0 + t\left(\dot{x}_0 + x_0\omega\right)\right]$

MOTELGARRITASUN AZPIKRITIKOA $\boxed{\xi = 0{,}25}$

$$\begin{cases} x(t) = e^{-\xi\omega t}\left[x_0 \cos(\omega_D t) + \left(\dfrac{\dot{x}_0 + \xi\omega x_0}{\omega_D}\right) \sin(\omega_D t)\right] \\ x(t) = e^{-\xi\omega t}\sqrt{x_0^2 + \left(\dfrac{\dot{x}_0 + \xi\omega x_0}{\omega_D}\right)^2} \cos\left(\omega_D t - arctg\,\dfrac{\dot{x}_0 + \xi\omega x_0}{\omega_D x_0}\right) \end{cases}$$



## INDAR HARMONIKOAK ERAGINDAKO BIBRAZIO BEHARTUAK




$$\boxed{m\ddot{x}+c\dot{x}+kx = f_0\cos(\omega_e t)}$$

$$\begin{cases} x(0)=x_0 \\ \dot{x}(0)=\dot{x}_0 \end{cases}$$

$$\boxed{x_h = e^{-\xi\omega t}\left(A\cos\omega_D t + B\sin\omega_D t\right)}$$

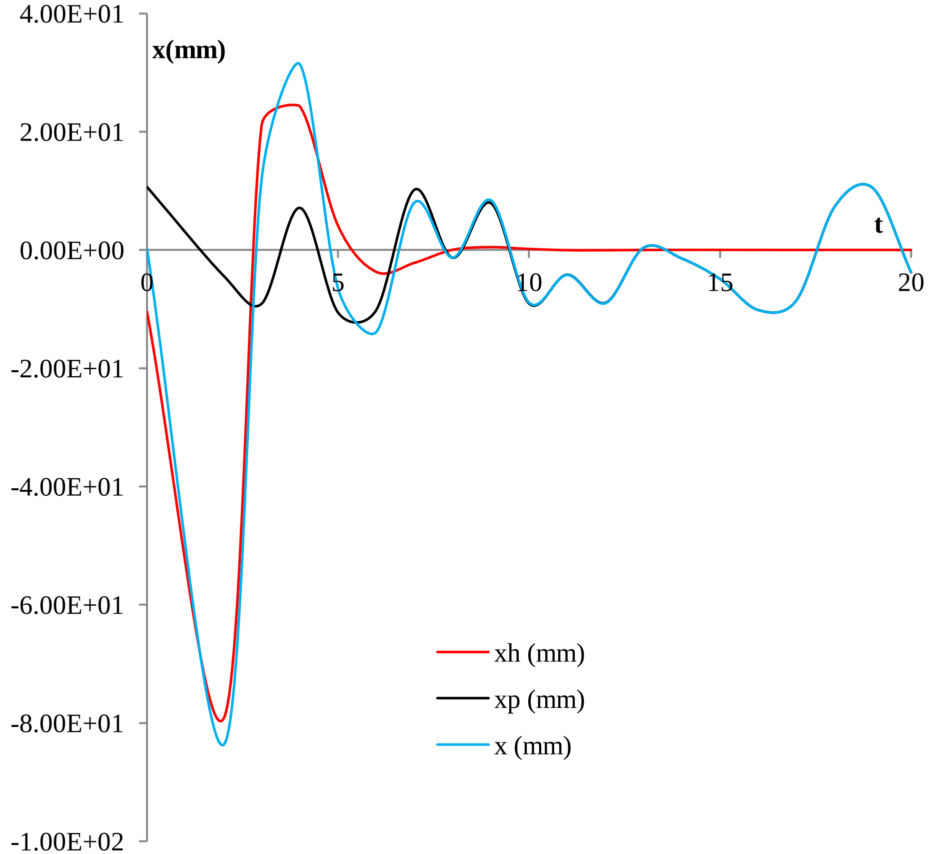
$$\begin{cases} A = x_0 - X_{est}D\cos\varphi \\ B = \dfrac{\dot{x}_0+\xi\omega x_0}{\omega_D} - X_{est}D\left[\dfrac{\xi\omega\cos\varphi+\omega_e\sin\varphi}{\omega_D}\right] \end{cases}$$

$$\boxed{x_p(t) = X_{est}D\cos(\omega_e t-\varphi)}$$

$$X_{est} = \frac{f_0}{k} \qquad tg\varphi = \frac{2\xi\beta}{1-\beta^2}$$

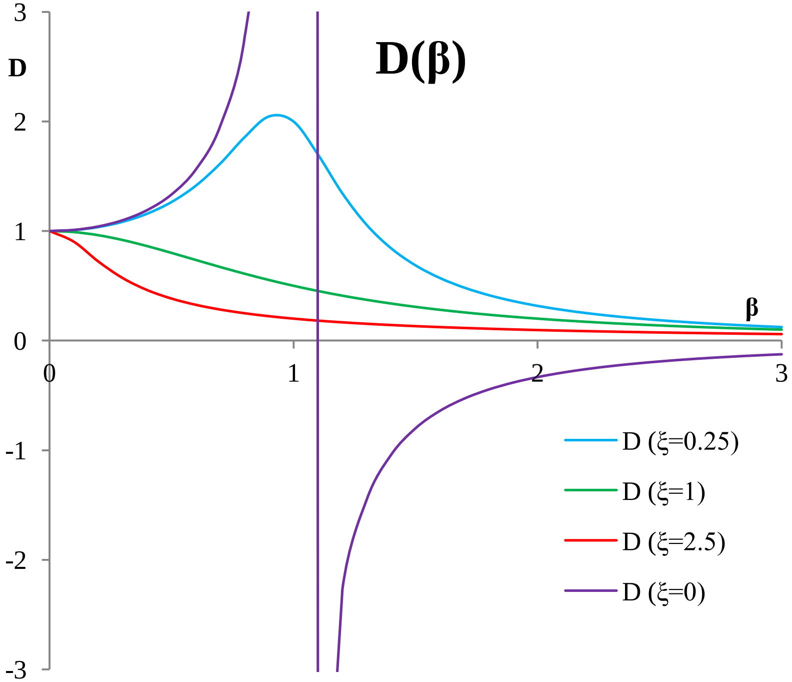
$$D = \frac{1}{\sqrt{(1-\beta^2)^2+(2\beta\xi)^2}} \qquad \beta = \frac{\omega_e}{\omega} \quad \xi = \frac{c}{2m\omega}$$

$$\boxed{x(t) = x_h + x_p} \longrightarrow \boxed{x(t) = e^{-\xi\omega t}\left(A\cos\omega_D t + B\sin\omega_D t\right) + X_{est}D\cos(\omega_e t-\varphi)}$$



## ANPLIFIKAZIO DINAMIKOA ETA DESFASEA

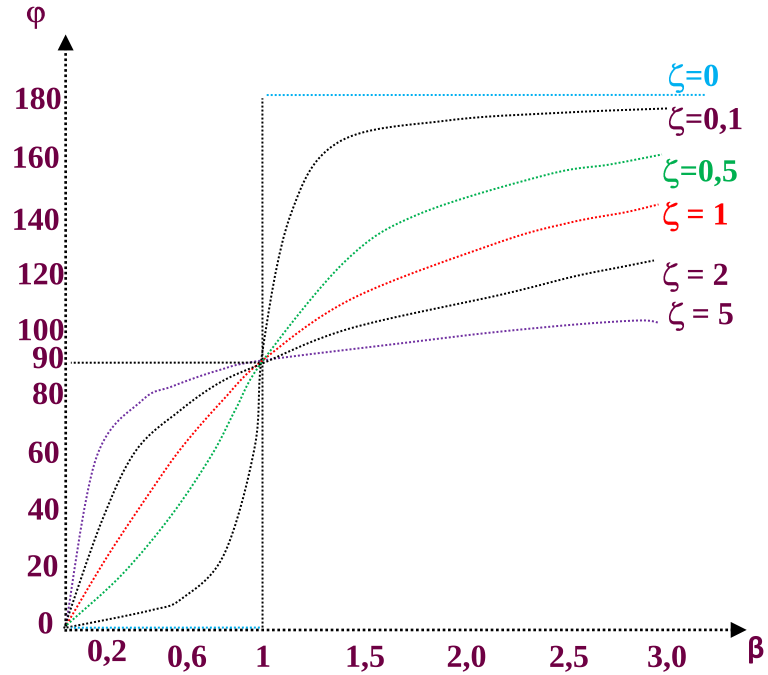
### D ANPLIFIKAZIO DINAMIKOA




$$\frac{dD}{d\beta} = 0 \longrightarrow \quad \beta = \sqrt{1-2\xi^2}$$

$$D_{\max} = D\left(\beta = \sqrt{1-2\xi^2}\right) = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

### φ DESFASEA



### MOTELDU GABEKO BIBRAZIO BEHARTUAK (ξ=0)

$$f(t) = f_0 \cos \omega_e t$$

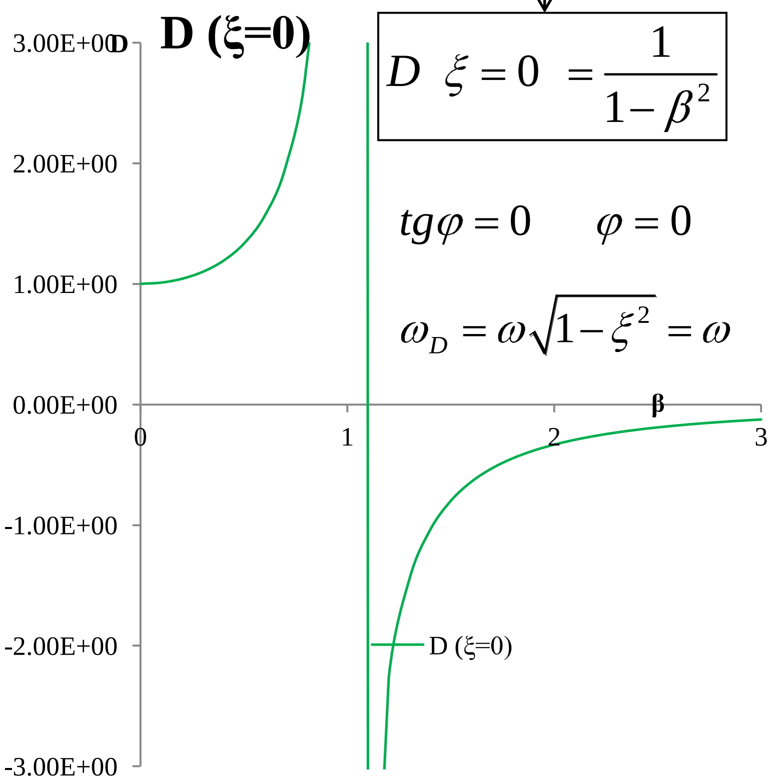
$$D = \frac{1}{\sqrt{\left(1-\beta^2\right)^2 + \left(2\beta\xi\right)^2}}$$

$$\xi = 0 \rightarrow \left\{ \begin{array}{l} x(t) = e^{-\xi\omega t}\left(A\cos\omega_D t + B\sin\omega_D t\right) + X_{est} D\cos\left(\omega_e t - \varphi\right) \\ \left.\begin{array}{l} x(0) = 0 \\ \dot{x}(0) = 0 \end{array}\right\} \rightarrow \left.\begin{array}{l} A = -X_{est} D \\ B = 0 \end{array}\right\} \rightarrow \boxed{x(t) = \frac{X_{est}}{1-\beta^2}\left(\cos\omega_e t - \cos\omega t\right)} \end{array}\right.$$

$$\boxed{D\left(\xi = 0\right) = \frac{1}{1-\beta^2}}$$

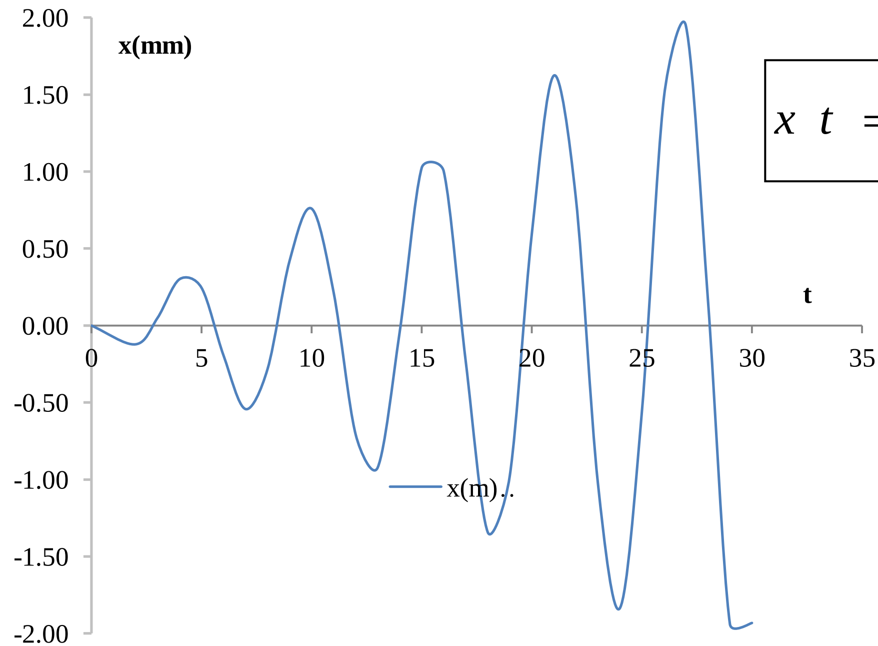
$$tg\varphi = 0 \qquad \varphi = 0$$

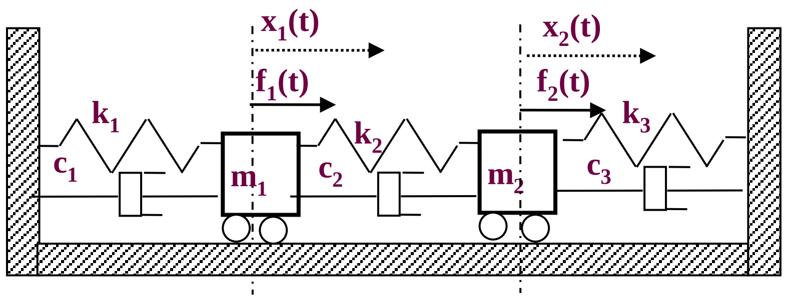
$$\omega_D = \omega\sqrt{1-\xi^2} = \omega$$




**ERRESONANTZI KASUAN (ωe= ω -> β = 1)**

$$\boxed{x(t) = \frac{X_{est}\,\omega}{2}\, t\cdot\sin\omega t}$$






$$\boxed{M\ \ddot{x} + C\ \dot{x} + K\ x = f(t)}$$

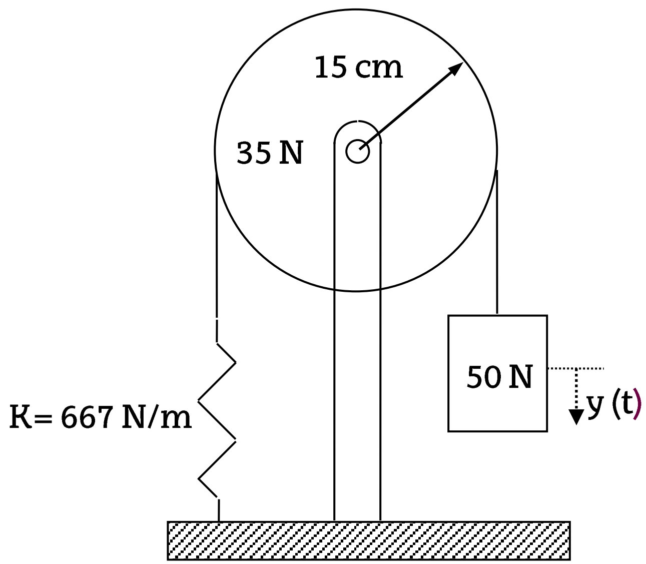
| | |
|---|---|
| $x$ | Desplazamenduen zutabe-matrizea |
| $M$ | Masa-matrizea edo Inertzia-matrizea |
| $C$ | Motelgarritasun-matrizea |
| $K$ | Zurruntasun-matrizea |
| $f(t)$ | Indarren zutabe-matrizea |

### PROBLEMA 1

Irudiko 50 N-eko blokea lotuta dago hari batekin. Haria 35N-eko zilindro uniformearen gainean biribilduta dago. Hariak ez badu irristatzen zilindroaren gainean:

a) Idatzi higidurako ekuazio diferentziala 50 N-eko blokearen y(t) posizioarentzat.

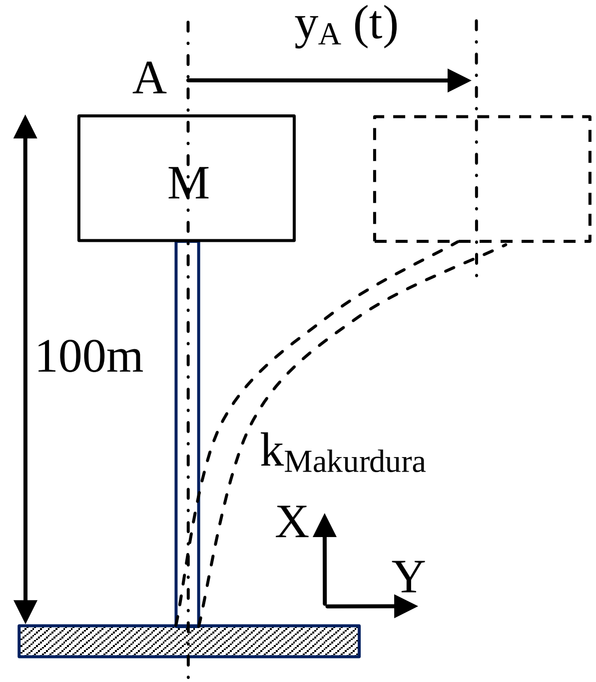
b) Kalkulatu higudura bibratoriaren frekuentzia eta periodoa.



## PROBLEMA 2

Izan bedi irudian agertzen den bezalako ur-tanke baten euskarri-egitura. Bere garaiera 100 m-koa da. Zutabeak sekzio zirkular hustua dauka, bere kanpo diametroa $d_0$ = 3m delarik, eta barnealdekoa $d_i$ = 2,5 m. Tankeak $3 \cdot 10^5$ kg ditu urez beteta. Tankearen makurdura bibrazioen maiztasun naturala kalkulatzea eskatzen da, zeharkako kitzikadura bat ezartzen zaionean (adibidez burrunbada sismiko bat).

Oharra: Sistemaren masa tankearena dela gutxi gorabehera zutabearen muturrean ezarrita kontuan hartuko da.

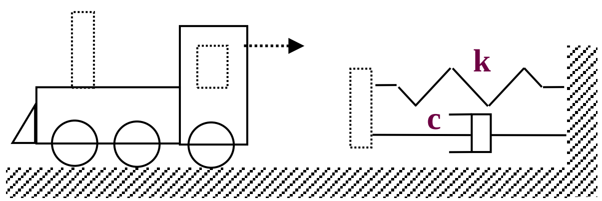



**Emaitzak:** $\omega$ = 2,079 $s^{-1}$

## PROBLEMA 3

2500 kg-ko lokomotora bat nasa batera heltzen da 10 km/h-ko abiaduarekin, gero helmugan tope batean kateatzeko. Tope hori malguki-motelgailu sistema batekin dago eginda, zurruntasuna k = 30 N/mm eta motelgarritasuna c = 35 Ns/mm izanik. Honako hauek kalkulatzea eskatzen da:

- Ibilgailuaren desplazamendu maximoa topearekin bat egin ondoren.
- Maximo horretara heltzeko behar duen denbora.




**EMAITZAK** $\boxed{x_{max} = 0{,}19\,m}$ $\boxed{t = 0{,}21 s}$

# 6.- BIBLIOGRAFIA

- Hernández, A.; Pinto, C.; Petuya V.; Agirrebeitia J. Bibrazioen teoria. Oinarrizko jakingarriak. Bilboko Ingeniaritza Goi Eskola Teknikoa, Bilbao, 2000