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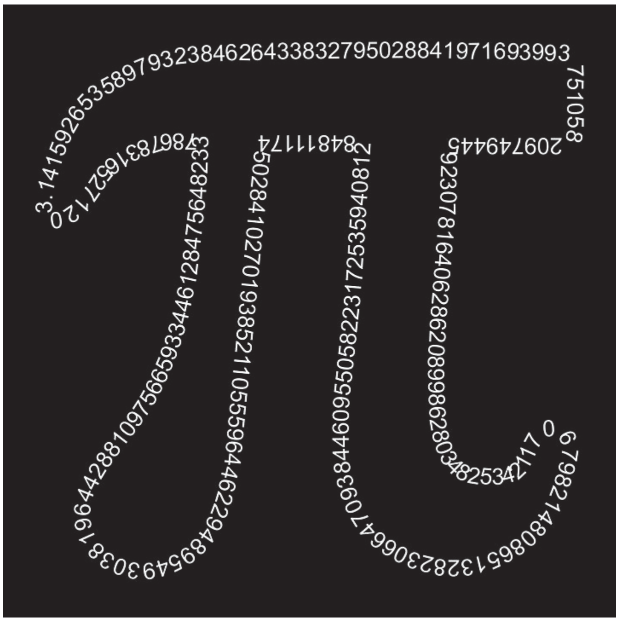
MATHS BASIC COURSE FOR UNDERGRADUATES



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TEST 2. SOLUTIONS

SOLUTION EXERCISE 1: For $n = 1$, $1^3 = 1$ and $\frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$, and the statement fulfills. Suppose now that the statement $1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$ fulfills and consider the case $k+1$. We have that $1^3+2^3+3^3+\cdots+k^3+(k+1)^3 = \frac{k^2(k+1)^2}{4}+(k+1)^3 = \frac{k^2(k+1)^2}{4}+\frac{4(k+1)^3}{4} = \frac{(k+1)^2}{4}[4(k+1)+k^2] = \frac{(k+1)^2}{4}(k+2)^2 = \frac{(k+1)^2((k+1)+1)^2}{4}$, and the statement holds also for the case $k+1$.

SOLUTION EXERCISE 2: $|z_1| = \sqrt{1+3} = \sqrt{4} = 2$, $\theta_1 = \arg z_1 = \text{arctag}\sqrt{3} = 60°$ and $|z_2| = \sqrt{2+2} = \sqrt{4} = 2$, $\theta_2 = \arg z_2 = \text{arctag}\frac{\sqrt{2}}{\sqrt{2}} = 45°$. Thus $z = z_1 z_2 = 2_{60°}2_{45°} = 4_{105°}$. The cubic roots of the complex number z are $\sqrt[3]{4}_{35°}$, $\sqrt[3]{4}_{155°}$ and $\sqrt[3]{4}_{275°}$.

SOLUTION EXERCISE 3: We calculate the correponding divisions.

$$2012 = 486.4 + 68$$

$$486 = 68.7 + 10$$

$$68 = 10.6 + 8$$

$$10 = 8.1 + 2$$

$$8 = 2.4 + 0$$

Thus, $\gcd(2012, 486) = \gcd(486, 68) = \gcd(68, 10) = \gcd(10, 8) = \gcd(8, 2) = 2$. In addition to this, $2 = 10 - 8.1 = 10 - [68 - 10.6] = (10 + 6.10) - 68 = 7.10 + (-68) = 7[486 - 68.7] + (-68) = 7(486) + (-49 - 1).68 = 7(486) + (-50).68 = 7(486) + (-50)[2012 - (486)4] = (7 + 200).486 + (-50).2012 = 207(486) + (-50).2012$.

SOLUTION EXERCISE 4: Let us write all the natural numbers between 150 and 219 in the following table:

150	151	152	153	154	155	156	157	158	159
160	161	162	163	164	165	166	167	168	169
170	171	172	173	174	175	176	177	178	179
180	181	182	183	184	185	186	187	188	189
190	191	192	193	194	195	196	197	198	199
200	201	202	203	204	205	206	207	208	209
210	211	212	213	214	215	216	217	218	219

We start crossing out all the multiples of the prime number 2, beginning from 150, i.e, we cross out the corresponding natural numbers every 2 steps (or i.e, all the even numbers), and we continue crossing out all the multiples of the prime number 3, beginning from 150, i.e, we cross out the corresponding natural numbers every 3 steps; after, we cross out all the multiples of the prime number 5, beginning again from 150, i.e, we cross out all the corresponding natural numbers every 5 steps; later, we do the same for the prime number 7, beginning from the number 154, which is a multiple of 7; and next we do the same for the prime number 11, beginning again from the number 154, which is also a multiple of 11. Finally, we cross out all the

multiples of the prime number 13, beginning from 156, which is a multiple of 13.
The process is finished when we erase or cross out all the multiples of the primes p, being $p \leq \sqrt{219}$. Obviously, a number could be crossed out several times.
All the remainder numbers in this process are prime numbers. Thus, in the following table, the prime numbers between 150 and 219 are the ones that are not crossed out.

~~150~~	151	~~152~~	~~153~~	~~154~~	~~155~~	~~156~~	157	~~158~~	~~159~~
~~160~~	~~161~~	~~162~~	163	~~164~~	~~165~~	~~166~~	167	~~168~~	~~169~~
~~170~~	~~171~~	~~172~~	173	~~174~~	~~175~~	~~176~~	~~177~~	~~178~~	179
~~180~~	181	~~182~~	~~183~~	~~184~~	~~185~~	~~186~~	~~187~~	~~188~~	~~189~~
~~190~~	191	~~192~~	193	~~194~~	~~195~~	~~196~~	197	~~198~~	199
~~200~~	~~201~~	~~202~~	~~203~~	~~204~~	~~205~~	~~206~~	~~207~~	~~208~~	~~209~~
~~210~~	211	~~212~~	~~213~~	~~214~~	~~215~~	~~216~~	~~217~~	~~218~~	~~219~~

In other words, the prime numbers between 150 and 219 are $151, 157, 163, 173, 179, 181, 191, 193, 197, 199$ and 211.

SOLUTION EXERCISE 5: First of all, we realize that the linear congruence $10x \equiv 3 (\text{mod } 23)$ has an unique solution, since $\gcd(10, 23) = 1 \mid 3$, and that the linear congruence $5x \equiv 4(\text{mod } 27)$ has an unique solution as well, since $\gcd(5, 27) = 1 \mid 4$. On the other hand, the inverse of 10 module 23 is 7, since $7.10 = 70 \equiv 1(\text{mod } 23)$, and the inverse of 5 module 27 is 11, since $5.11 = 55 \equiv 1(\text{mod } 27)$. Thus, multiplying by 7 the linear congruence $10x \equiv 3(\text{mod } 23)$ we have that $x \equiv 3.7 = 21(\text{mod } 23)$, and multiplying by 11 the linear congruence $5x \equiv 4(\text{mod } 27)$ we have that $x \equiv 4.11 = 44 \equiv 17(\text{mod } 27)$.
Thus, solving the linear congruence system:

$$\begin{array}{rcll} 10x & \equiv & 3 & (\text{mod } 23) \\ 5x & \equiv & 4 & (\text{mod } 27) \end{array}$$

is equivalent to solving the linear congruence system:

$$\begin{array}{rcll} x & \equiv & 21 & (\text{mod } 23) \\ x & \equiv & 17 & (\text{mod } 27); \end{array}$$

and if x_1 is a particular solution of that congruence system, then $x \equiv x_1(\text{mod lcm}(23, 27))$, i.e, $x \equiv x_1(\text{mod } 621)$ is also a solution of the same congruence system. Thus, it remains to us to find a particular solution x_1 of the initial congruence system. To do this, let us consider $x = 21 + 23k$, for some integer k, and replace it on the second linear congruence. Then $21 + 23k \equiv 17(\text{mod } 27)$, i.e, $23k \equiv -4(\text{mod } 27)$, i.e, $23k \equiv 23(\text{mod } 27)$. So we could take $k = 1$ and we could consider $x_1 = 21 + 23.1 = 44$.

SOLUTION EXERCISE 6: Fermat's Little Theorem: If p is a prime number and $a \in \mathbb{Z}$ such that $p \nmid a$, then $a^{p-1} \equiv 1(\text{mod } p)$. Thus, applying the previous Theorem to the prime p and the integer numbers $1, 2, \ldots, p-1$ satisfying that neither of them is a multiple of p, we have that

$$1^{p-1} \equiv 1(\text{mod } p)$$

$$2^{p-1} \equiv 1(\text{mod } p)$$
$$\vdots$$
$$(p-1)^{p-1} \equiv 1(\text{mod } p).$$

Therefore, $1^{p-1} + 2^{p-1} + \cdots + (p-1)^{p-1} \equiv 1 + 1 + \cdots \overset{(p-1)\text{ -times}}{\cdots} + 1 \equiv p - 1(\text{mod } p) \equiv -1(\text{mod } p)$. In conclusion, $1^{p-1} + 2^{p-1} + \cdots + (p-1)^{p-1} \equiv -1(\text{mod } p)$, as required.

SOLUTION EXERCISE 7: (i) Being $p(x) = x^4 + 2x^3 - x^2 - 2x + 1$, $p'(x) = 4x^3 + 6x^2 - 2x - 2$. Applying the division algorithm to the polynomials $p(x)$ and $p'(x)$ we have that,

$$x^4 + 2x^3 - x^2 - 2x + 1 = (4x^3 + 6x^2 - 2x - 2)(\frac{1}{4}x + \frac{1}{8}) + (\frac{-5}{4}(x^2 + x - 1));$$
$$4x^3 + 6x^2 - 2x - 2 = -\frac{5}{4}(x^2 + x - 1)[(\frac{-4}{5})2(1 + 2x)] + 0.$$

Thus, $\gcd(p(x), p'(x)) = \gcd(4x^3 + 6x^2 - 2x - 2, \frac{-5}{4}(x^2 + x - 1)) = \frac{-5}{4}(x^2 + x - 1) \sim (x^2 + x - 1)$.
(ii) If $a \in \mathbb{R}$ would be a multiple root of the polynomial $p(x)$, then a would be a common root of the polynomials $p(x)$ and $p'(x)$, in other words, a would be a root of the $\gcd(p(x), p'(x)) = x^2 + x - 1$. Thus, let us calculate the roots of the polynomial $x^2 + x - 1$. These are $\alpha_1 = \frac{-1+\sqrt{5}}{2}$ and $\alpha_2 = \frac{-1-\sqrt{5}}{2}$. In consequence, $(x - \alpha_1)^2(x - \alpha_2)^2 = ((x - \alpha_1)(x - \alpha_2))^2 \mid p(x)$. On the other hand, since $(x - \alpha_1)(x - \alpha_2) = x^2 + x - 1$, it follows that $(x^2 + x - 1)^2 \mid p(x)$. Finally, since $p(x)$ is a monic polynomial of degree 4 and also $(x^2 + x - 1)^2$ is a polynomial of degree 4, we conclude that $p(x) = (x^2 + x - 1)^2$.

SOLUTION EXERCISE 8: First of all, we decompose the polynomial $x^4 + 2x^3 + x^2$ as $x^4 + 2x^3 + x^2 = x^2(x + 1)^2$. We propose $\frac{3x+1}{x^4+2x^3+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)} + \frac{D}{(x+1)^2}$. Making computations we have that,

$$\frac{3x + 1}{x^4 + 2x^3 + x^2} = \frac{Ax(x + 1)^2 + B(x + 1)^2 + Cx^2(x + 1) + Dx^2}{x^2(x + 1)^2}.$$

In particular, $3x+1 = Ax(x + 1)^2 + B(x + 1)^2 + Cx^2(x+1) + Dx^2$. Evaluating both expressions on some values:

$$x = 0, \text{ we have that } 1 = B$$
$$x = -1, \text{ we have that } -2 = D \Longrightarrow D = -2$$
$$x = 1, \text{ we have that } 2A + C = 1 \Longrightarrow C = 1 - 2A$$
$$x = 2, \text{ we have that } 3A + 2C = 1.$$

Substituting C in the expression $3A+2C = 1$, it follows that $3A+2(1-2A) = -A + 2 = 1$, and consequently $A = 1$ and thus $C = -1$.
Therefore $\frac{3x+1}{x^4+2x^3+x^2} = \frac{1}{x} + \frac{1}{x^2} - \frac{1}{(x+1)} - \frac{2}{(x+1)^2}$.

SOLUTION EXERCISE 9: First of all, observe that

$$|x - 1| = \begin{cases} -x + 1, & \text{if } x < 1 \\ x - 1, & \text{if } x \geq 1 \end{cases}$$

$$|x+4| = \begin{cases} -x-4, & \text{if} \quad x < -4 \\ x+4, & \text{if} \quad x \geq 4 \end{cases}$$

Thus, if $x < -4$ we propose $-x+1-x-4 > 10$, which implies $-2x-3 > 10$ i.e. $2x+3 < -10$ i.e. $x < -\frac{13}{2}$.
If $-4 \leq x < 1$ we propose $-x+1+x+4 > 10$, which implies $5 > 10$. This means that there is not solution of the initial inequation for $-4 \leq x < 1$. Finally, if $x \geq 1$, we propose $x-1+x+4 > 10$, which implies $2x > 7$ i.e $x > \frac{7}{2}$.
Thus, the solution of the initial inequation is $(-\infty, -\frac{13}{2}) \cup (\frac{7}{2}, \infty)$.