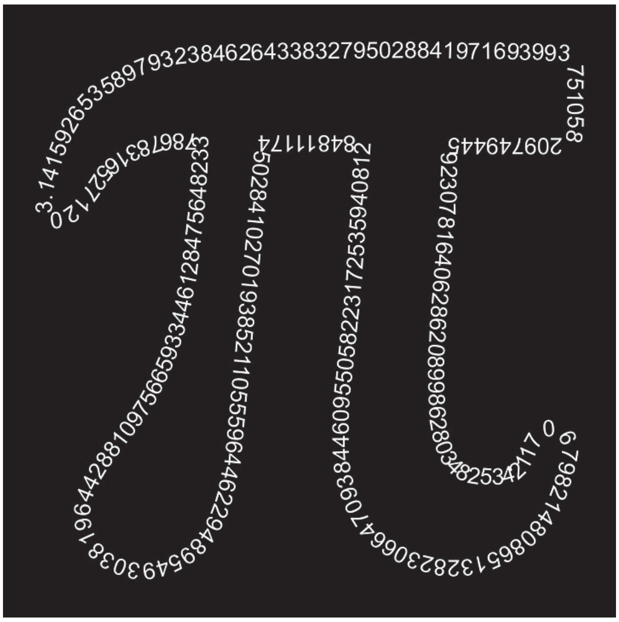
MATHS BASIC COURSE FOR UNDERGRADUATES



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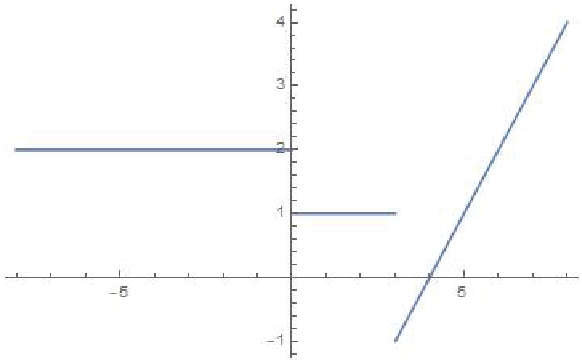
TEST 1. SOLUTIONS

SOLUTION EXERCISE 1: If $n = 3$, $2^3 = 8 > 2.3 + 1 = 7$, and the statement fulfills. Suppose now that for any $k > 3$ the statement $2^k > 2k+1$ fulfills, and we prove that the same happens for the case $k + 1$. In fact, $2^{k+1} = 2^k.2 > (2k+1).2 = 4k+2 = 2k+2+2k > 2(k+1)+1$, since $2k > 1$.

SOLUTION EXERCISE 2: (i) Clearly f is not injective, since for instance $f(-1) = 2 = f(-2)$, and $-1 \neq -2$. Besides, f is not surjective, since $f(\mathbb{R}) = (-1, \infty) \neq (-\infty, \infty)$.

(ii) Taking into account the graph of the function f, we realize that $f((1, 3)) = \{1\}$ and $f^{-1}((0, 1)) = (4, 5)$.

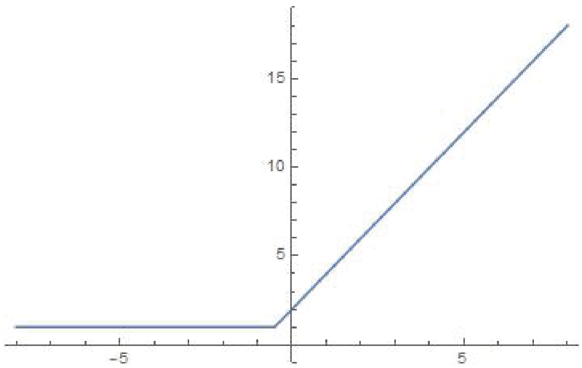


SOLUTION EXERCISE 3:

(i) First we compute the function $f \circ g$ as follows:

$$(f \circ g)(x) = \begin{cases} 1, & \text{when} \quad x \leq -1/2 \\ 2x + 2, & \text{when} \quad x > -1/2 \end{cases}$$

Clearly $f \circ g$ is not injective, since $(f \circ g)(-1) = 1 = (f \circ g)(-2)$, and $-1 \neq -2$. Neither the function $f \circ g$ is surjective, since $(f \circ g)(\mathbb{R}) \neq (-\infty, \infty)$. Below the graph of the function $f \circ g$ appears:

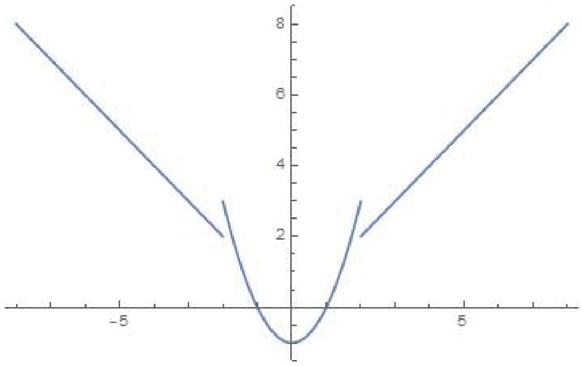


(ii) Taking into account the graph of the function $f \circ g$, we observe that $(f \circ g)^{-1}(1) = (-\infty, -1/2]$ and that $(f \circ g)^{-1}(2) = \{0\}$.

SOLUTION EXERCISE 4:

(i) Since $f(-1) = 0 = f(1)$, it follows that f is not injective. Besides, $f(\mathbb{R}) = [-1, \infty)$, which means that f is not surjective.

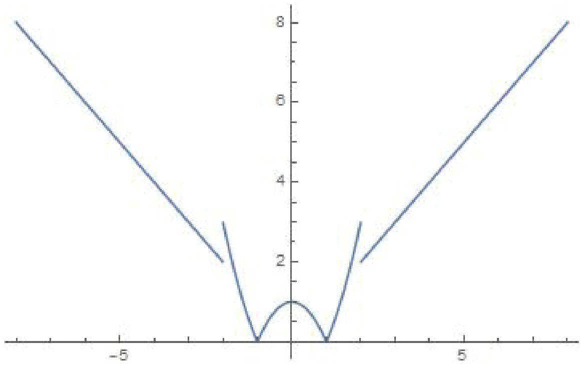
(ii) Taking into account the graph of f, we realize that $f((0,3]) = (-1,3]$ and $f^{-1}([-3,3]) = [-3,-2) \cup [-2,2] \cup (2,3] = [-3,3]$.



(iii) The composition function $g \circ f$ is computed as follows:

$$(g \circ f)(x) = \begin{cases} -x, & \text{if } \; x < -2 \\ x^2 - 1, & \text{if } \; -2 \leq x \leq -1 \\ -x^2 + 1, & \text{if } \; -1 < x < 1 \\ x^2 - 1, & \text{if } \; 1 \leq x \leq 2 \\ x, & \text{if } \; x > 2 \end{cases}$$

Below it appears the graph of the function $g \circ f$.



Finally, $(g \circ f)^{-1}(1) = \{-\sqrt{2}, \sqrt{2}, 0\}$.

SOLUTION EXERCISE 5: Take any $z_0 \in Z$. Since the function $g \circ f : X \longrightarrow Z$ is surjective, there exists some $x_0 \in X$ such that $(g \circ f)(x_0) = z_0$. In particular, $(g \circ f)(x_0) = g(f(x_0)) = z_0$, being $f(x_0)$ an element of Y. It means, that there exists $y_0 = f(x_0) \in Y$ such that $g(y_0) = z_0$, i.e. g is a surjective function. However, the converse does not always hold. Consider, for instance $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = |x|$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ such that $g(x) = x + 1$. It is easy to prove that g is surjective, but $g \circ f : \mathbb{R} \longrightarrow \mathbb{R}$ defining as $(g \circ f)(x) = |x| + 1$ is not surjective.

SOLUTION EXERCISE 6: First of all, we can suppose that $n \in \mathbb{N}$. Taking into account the remainders module 5, n can be written as $5q_1$, $5q_2 + 1$, $5q_3 + 2$, $5q_4 + 3$ or $5q_5 + 4$, for some $q_1, q_2, q_3, q_4, q_5 \in \mathbb{N}$. Thus, computing the square of n we have

$$n^2 = (5q_1)^2 = 25{q_1}^2 = 5(5q_1^2)$$

$$n^2 = (5q_2 + 1)^2 = 25q_2^2 + 10q_2 + 1 = 5(5q_2^2 + 2q_2) + 1$$

$$n^2 = (5q_3 + 2)^2 = 25q_3^2 + 20q_3 + 4 = 5(5q_3^2 + 4q_3) + 4$$
$$n^2 = (5q_4 + 3)^2 = 25q_4^2 + 30q_4 + 9 = 5(5q_4^2 + 6q_4 + 1) + 4$$
$$n^2 = (5q_5 + 4)^2 = 25q_5^2 + 40q_5 + 16 = 5(5q_5^2 + 8q_5 + 3) + 1,$$

where $5q_1^2$, $5q_2^2 + 2q_2$, $5q_3^2 + 4q_3$, $5q_4^2 + 6q_4 + 1$ and $5q_5^2 + 8q_5 + 3$ are natural numbers.

SOLUTION EXERCISE 7: (i) Let us denote by $d_1 = \gcd(a, b)$ and by $d_2 = \gcd(b, r)$. By the first property of d_1 we have that $d_1 \mid a$ and $d_1 \mid b$. In particular, $d_1 \mid bq$, and consequently $d_1 \mid (a - bq) = r$. Now using the second property of d_2, it follows that $d_1 \mid d_2$. On the other hand, by the first property of d_2, we have that $d_2 \mid b$ and $d_2 \mid r$. In particular, $d_2 \mid bq$ and consequently $d_2 \mid (bq + r) = a$. Now using the second property of d_1, it follows that $d_2 \mid d_1$. Finally, since $d_1 \mid d_2$, $d_2 \mid d_1$ and $d_1, d_2 \in \mathbb{N}$, we conclude that $d_1 = d_2$, as required.

(ii) Making calculations we have that

$$102 = 44.2 + 14$$
$$44 = 14.3 + 2$$
$$14 = 2.7 + 0.$$

Applying the previous item (i), it follows that $\gcd(102, 44) = \gcd(44, 14) = \gcd(14, 2) = 2$. Now making substitutions we have that $2 = 44 - 14.3 = 44 - 3.(102 - 44.2) = 44 + 3.2.44 + (-3).102 = 7.44 + (-3).102$. In conclusion $(7, -3) \in \mathbb{Z} \times \mathbb{Z}$ is a solution for the equation $44x + 102y = 2$.

SOLUTION EXERCISE 8: (i) Let us denote by $d_1 = \gcd(ac, b)$ and by $d_2 = \gcd(c, b)$. By the first property of d_1, we have that $d_1 \mid ac$ and $d_1 \mid b$. Since $\gcd(a, b) = 1$, being d_1 a divisor of b, d_1 should not be a divisor of a, unless $d_1 = 1$, and as well $\gcd(d_1, a) = 1$. Now since $d_1 \mid ac$ and $\gcd(d_1, a) = 1$, it follows that $d_1 \mid c$, and therefore, using the second property of d_2, it follows that $d_1 \mid d_2$. On the other hand, by the first property of d_2, we have that $d_2 \mid c$ and $d_2 \mid b$. In particular, $d_2 \mid ac$, and by the second property of d_1, it follows that $d_2 \mid d_1$. Finally, since $d_1 \mid d_2$, $d_2 \mid d_1$ and $d_1, d_2 \in \mathbb{N}$, we conclude that $d_1 = d_2$, as required.

(ii) We observe that $\gcd(5000, 31768) = \gcd(31768, 5000)$. Besides $31768 = 11.19^2.8$ and the number 11.19^2 is coprime with 5000. So using the previous item (i) we get that $\gcd(31768, 5000) = \gcd(8, 5000) = 8$, since $5000 = 8.5^4$.