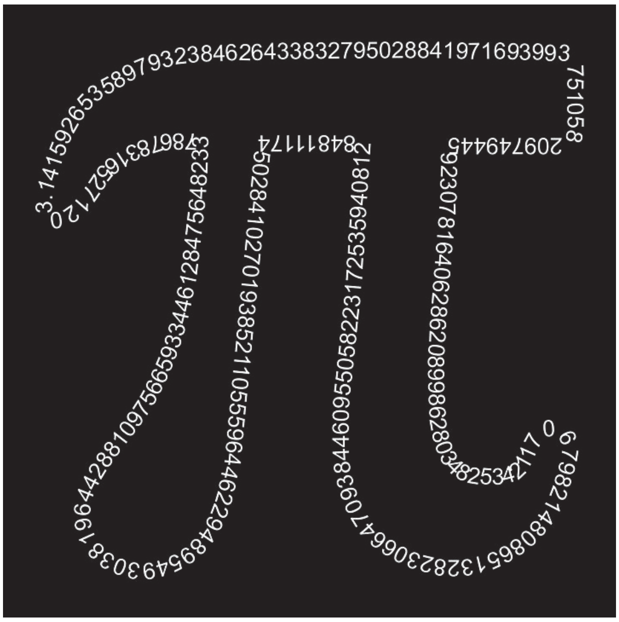
MATHS BASIC COURSE FOR UNDERGRADUATES



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SOLUTIONS: <u>8th SUBJECT. ELEMENTARY LINEAR ALGEBRA</u>

SOLUTION EXERCISE 1: We will prove that the set $B = \{(1,0,\ldots,0),(0,1,\ldots,0),\ldots,(0,0,\ldots,1)\}$ is a basis of K^n. (It is called the canonical basis of K^n). For i in $\{1,\ldots,n\}$, let us consider $e_i = (0,\ldots,0,1,0,\ldots,0)$, the vector with 1 in the i-th coordinate and 0 elsewhere. If $v = (a_1,\ldots,a_n) \in K^n$, then it is obvious that $v = a_1e_1 + \cdots + a_ne_n$. This proves that B generates K^n. Let us suppose that $\lambda_1 e_1 + \cdots + \lambda_n e_n = (0,\ldots,0)$. Then, $(\lambda_1,\ldots,\lambda_n) = (0,\ldots,0)$, and therefore $\lambda_i = 0$ for all i. This proves that B is free. Since B is free and generates K^n, then it is a basis of K^n. Finally, as the cardinality of B is n, it is deduced that the dimension of K^n is n.

SOLUTION EXERCISE 2: Subtracting the first column from the fourth column, we get that the determinant equals to

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 \\ 0 & -8 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{vmatrix},$$

and developing the determinant by the elements of the first row we get that the determinant is

$$\begin{vmatrix} 2 & 0 & 2 \\ -8 & 2 & 0 \\ 2 & 0 & 1 \end{vmatrix}.$$

By subtracting the first column from the third column we obtain that this is equals to

$$\begin{vmatrix} 2 & 0 & 0 \\ -8 & 2 & 8 \\ 2 & 0 & -1 \end{vmatrix},$$

and developing this determinant by the elements of the first row we get,

$$2\begin{vmatrix} 2 & 8 \\ 0 & -1 \end{vmatrix},$$

which is equal to $2(-2) = -4$.

SOLUTION EXERCISE 3:

(i) We prove it by induction on n. For $n = 1$, both expressions are 1. Let us assume that the statement is true for $n-1$. By subtracting (beginning from the last row) to each row the previous row multiplied by a_1 we get,

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix} = \begin{vmatrix} 1 & 1 & \dots & 1 \\ 0 & a_2 - a_1 & \dots & a_n - a_1 \\ \vdots & \vdots & & \vdots \\ 0 & a_2^{n-1} - a_1 a_2^{n-2} & \dots & a_n^{n-1} - a_1 a_n^{n-2} \end{vmatrix}.$$

By developing the determinant by the elements of the first column, and then factoring out, for each i, the element $a_{i+1} - a_1$ from the i-th column in the resulting $(n-1) \times (n-1)$ determinant, we obtain that the determinant corresponds to

$$(a_2 - a_1) \cdot \dots \cdot (a_n - a_1) \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_2 & a_3 & \dots & a_n \\ \vdots & \vdots & & \vdots \\ a_2^{n-2} & a_3^{n-2} & \dots & a_n^{n-2} \end{vmatrix},$$

and now the result follows from the induction hypothesis.

(ii) Factoring out a_j^i in the j-th column for each i, we get that the requested determinant is $(a_1 \cdot \dots \cdot a_n)^i \prod_{i>j} (a_i - a_j)$.

SOLUTION EXERCISE 4:
The adjoint matrix is

$$\begin{pmatrix} -4 & 6 & 1 \\ 2 & 0 & -2 \\ -2 & -6 & 2 \end{pmatrix},$$

and the transpose of the adjoint matrix is

$$\begin{pmatrix} -4 & 2 & -2 \\ 6 & 0 & -6 \\ 1 & -2 & 2 \end{pmatrix}.$$

The determinant of A is -6, and therefore the inverse of A is

$$\begin{pmatrix} 2/3 & -1/3 & 1/3 \\ 1 & 0 & 1 \\ -1/6 & 1/3 & -1/3 \end{pmatrix}.$$