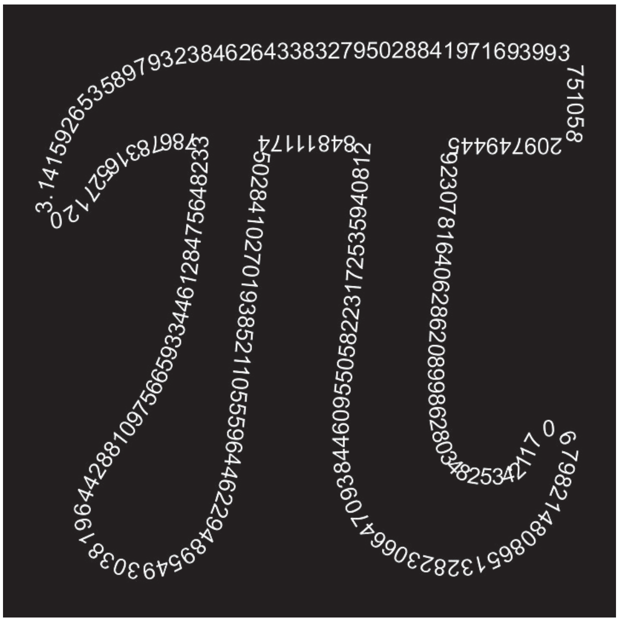
MATHS BASIC COURSE FOR UNDERGRADUATES



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SOLUTIONS: <u>6th SUBJECT. POLYNOMIALS</u>

SOLUTION EXERCISE 1: Observe that $f(x) = x^3 - 1 = (x-1)(x^2+x+1)$, and that the polynomial x^2+x+1 does not have roots over $\mathbb{R}$. Thus, the only real root of $f(x)$ is 1.

SOLUTION EXERCISE 2. Hint: $\gcd(x^5-1, x^3+x-2) = \gcd(x^3+x-2, 2x^2+x-3) = \gcd(2x^2+x-3, \frac{11x}{4} - \frac{11}{4}) = \gcd(\frac{11x}{4} - \frac{11}{4}, 0)$.
Thus, $\gcd(x^5-1, x^3+x-2) = x-1$.

SOLUTION EXERCISE 3. Hint: Argue by way of contradiction in both implications, taking into account that $\mathrm{dg}(f(x)) = 2$ or 3.

SOLUTION EXERCISE 4. Hint: Assume that a rational number $\frac{r}{s}$ is a root of $f(x)$. Then, the statement $f(\frac{r}{s}) = 0$ fulfills. Expand this expression and obtain from it, which are the conditions the integer numbers r and s must hold.

SOLUTION EXERCISE 5: If the polynomial $f(x) = 2x^3 - x^2 + 8x + 1 \in \mathbb{Z}[x]$ had a rational root, using the previous Exercise 4, it would be one of the possible values: $1, -1, 1/2$ and/or $-1/2$. However, $f(1) \neq 0$, $f(-1) \neq 0$, $f(1/2) \neq 0$ and $f(-1/2) \neq 0$. This means that $f(x)$ does not have rational roots.

SOLUTION EXERCISE 6: First of all, using the method given in Exercise 4, observe that the given polynomial $f(x)$ does not have any rational roots. On the other hand, if this polynomial $f(x)$ admitted a decomposition as a product of two polynomials of degree 2 with coefficients in $\mathbb{Z}$, then $f(x)$ will be expressed as $f(x) = (x^2+ax+b)(x^2+cx+d)$, satisfying the following conditions:

$$bd = 1, bc + da = 8, d + b + ac = -2 \text{ and } a + c = 0.$$

However, the previous system is incompatible, and consequently we conclude that $f(x)$ is irreducible over the field $\mathbb{Q}$.

SOLUTION EXERCISE 7: Let us consider the polynomial $f(x) = x^6 - 25x^5 + 3x^2 + 12 \in \mathbb{Z}[x]$ and the prime $p = 3$. Observe that $p = 3$ satisfies the following properties:

(i) $3 \mid 12, 3 \mid 0, 3 \mid 3, 3 \mid 0, 3 \mid 0$,

(ii) $9 \nmid 12$

(iii) $3 \nmid -25$.

Then, by Eisenstein's extended criterion we conclude that the polynomial $f(x)$ admits in $\mathbb{Z}[x]$ an irreducible factor of degree 5 or 6. In the former case, if $f(x)$ admits an irreducible factor of degree 5, then $f(x)$ would also admits a rational root (and this does not happen; it can be proved easily). Thus, $f(x)$ admits an irreducible factor of degree 6 in $\mathbb{Z}[x]$, and this means that $f(x)$ is irreducible over the field $\mathbb{Q}$.