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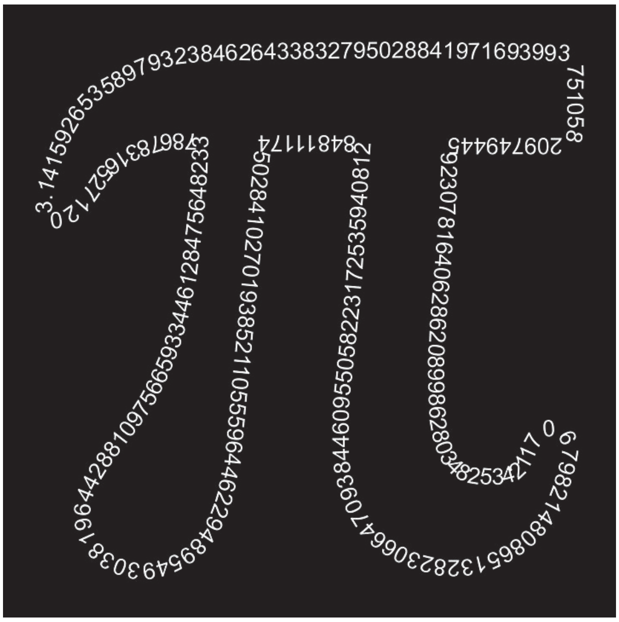
MATHS BASIC COURSE FOR UNDERGRADUATES



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SOLUTIONS: <u>3rd SUBJECT. FUNCTIONS</u>

SOLUTION EXERCISE 1: We have that, $\text{im}f = [0, +\infty)$, $f([0, 2]) = [0, 4]$, $f([2, +\infty)) = [4, +\infty)$, $f((-\infty, -1) \cup [2, +\infty)) = (1, +\infty)$, $f^{-1}(1) = \{-1, 1\}$, $f^{-1}(-1) = \emptyset$, $f^{-1}([-1, 0]) = \{0\}$ and $f^{-1}((1, +\infty)) = (-\infty, -1) \cup (1, +\infty)$.

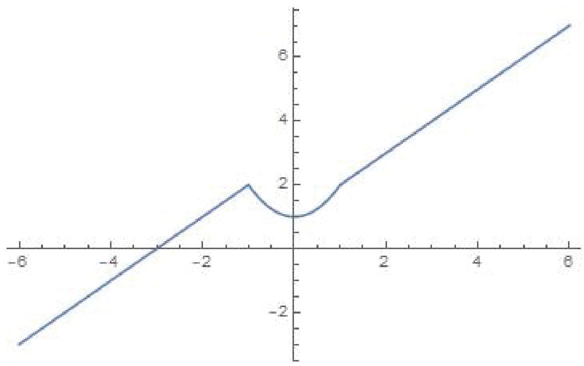
SOLUTION EXERCISE 2:

(i) If we consider the function $f : \mathbb{R} \longrightarrow [0, +\infty)$ given by $f(x) = x^2$, for any $x \in \mathbb{R}$, then it is clear that f is not a bijective map (it is surjective but it is not injective), and consequently the inverse map f^{-1} does not exist.

(ii) Consider now the function $g : [0, +\infty) \longrightarrow [0, +\infty)$ given by $g(x) = x^2$, for any $x \in \mathbb{R}$. We have that g is a restriction of the function f to the set $[0, +\infty)$, and clearly g is a bijective function. It means that the inverse function $g^{-1} : [0, +\infty) \longrightarrow [0, +\infty)$ exists. Let us calculate now which is the expression determined by g^{-1}. Remind that $g^{-1}(y) = x$ if and only if $g(x) = y$. Then, in order to calculate the image of any $y \in [0, +\infty)$ through the function g^{-1}, we should solve the equation $g(x) = y$, for the indeterminate x. We did the same when we analyzed the function g being surjective. We get $x = \sqrt{y}$. Thus, $g^{-1}(y) = \sqrt{y}$, and renaming the indeterminate, we have that $g^{-1}(x) = \sqrt{x}$, for any $x \in [0, +\infty)$.

(iii) Let $h : (-\infty, 0] \longrightarrow [0, +\infty)$ be the function given by the expression $h(x) = x^2$, for any $x \in \mathbb{R}$. In other words, let us consider the restriction of f to $(-\infty, 0]$. Repeating the same argument followed in the previous item (ii), we conclude that h is a bijective function, and it can be proved easily that $h^{-1}(x) = -\sqrt{x}$, for any $x \in [0, -\infty)$.

SOLUTION EXERCISE 3: (i) Clearly f is not injective. For instance, $f(-1) = 2 = f(1)$, and $-1 \neq 1$. Observing what the graph of the function f is, we conclude that f is surjective.



(ii) $f([0,2]) = [1,3]$ and $f^{-1}([0,2]) = [-3,1]$
(iii)

$$(f \circ g)(x) = \begin{cases} 1, & \text{when} \quad x = 0 \\ |x| + 2, & \text{when} \quad x \neq 0 \end{cases}$$

Finally, $(f \circ g)(\mathbb{R}) = \{1\} \cup [2, \infty)$.

SOLUTION EXERCISE 4: The obtained composition functions are the following:

$$(f \circ f)(x) = f(f(x)) = f(x^2) = (x^2)^2 = x^4,$$
$$(f \circ g)(x) = f(g(x)) = f(x+2) = (x+2)^2 = x^2 + 4x + 4,$$
$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 2,$$
$$(g \circ g)(x) = g(g(x)) = g(x+2) = (x+2) + 2 = x + 4.$$

Notice that $f \circ g \neq g \circ f$. Thus, the order of the composition factors is relevant.