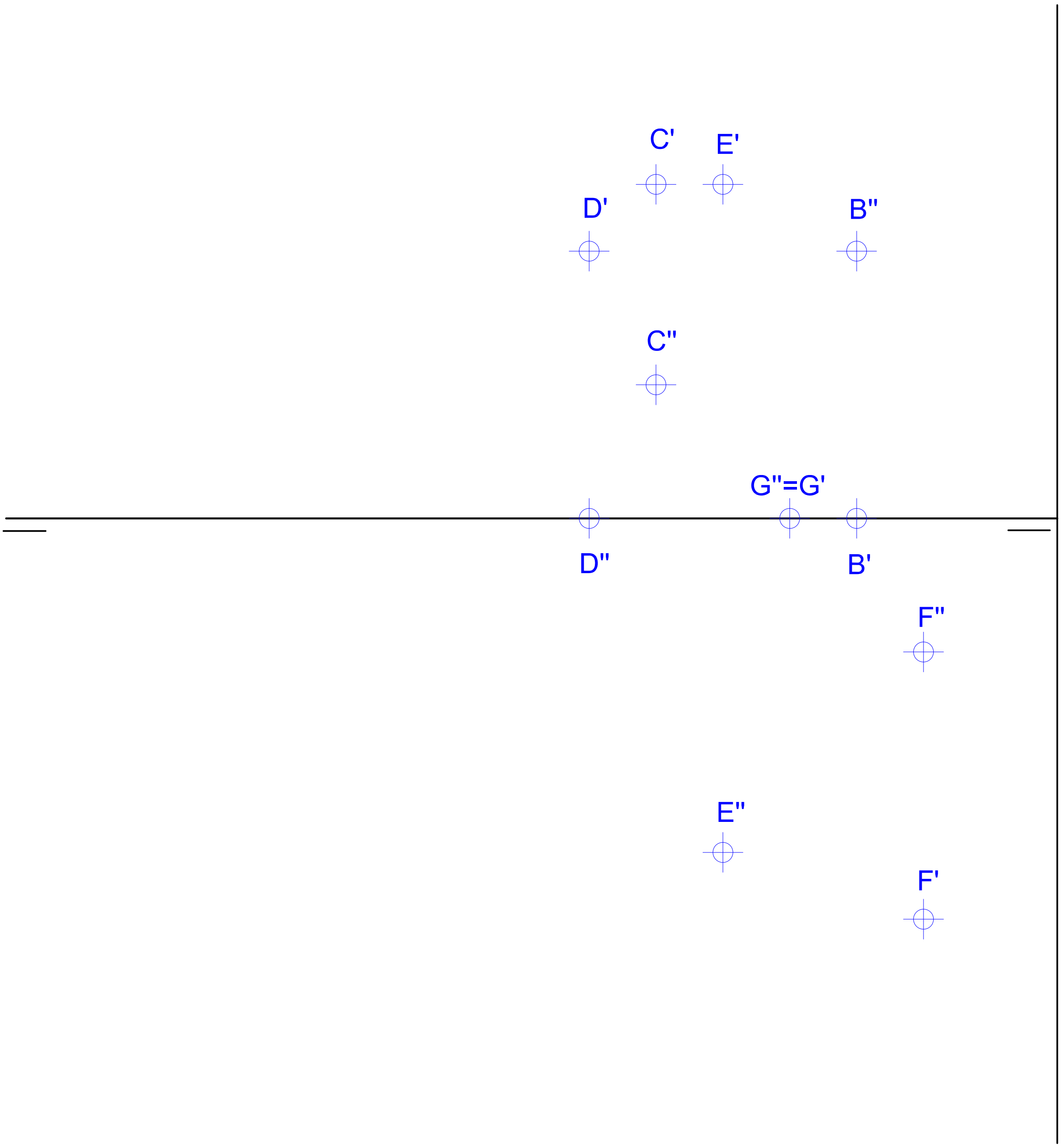
EXERCISE 1

Represent the diedric projections of the points B, C, D, E, F and G.



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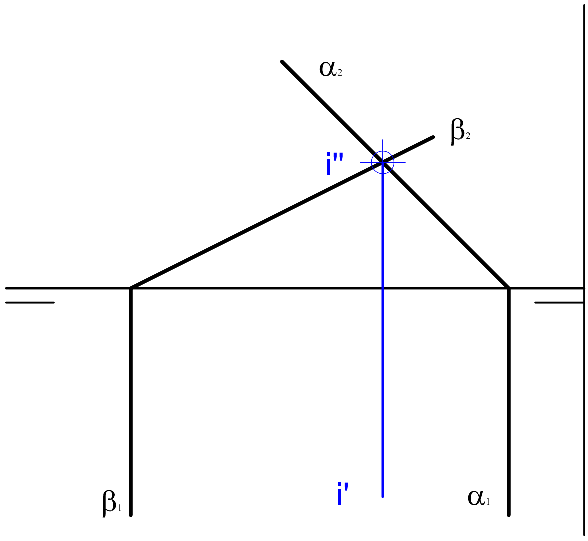
EXERCISE 2

Find the intersection of the plane α *determined by the points* $(4,0,3)$, $(1,0,0)$ *and* $(1,1,0)$, *and the plane* β *determined by* $(2,0,2)$, $(6,0,0)$ *and* $(6,3,0)$.

Find the intersection between the planes α and β.



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EXERCISE 3

Calculate the intersection between the planes β and α. β contains the points A=(6,3,1), B=(1,1,2) and C=(3,y,4), and it is perpendicular to the plane XOY. α contains the point P(1,1,2), and it is parallel to the plane XOZ.

Solution:

The procedure to calculate the intersection between the planes α and β is the following:

- Compute the plane β:

We obtain the normal vector of the plane doing the vector product of the vectors $\overrightarrow{AB} = (-5,-2,1)$ and $\overrightarrow{AC} = (-3, y-3,3)$:

$$\vec{n} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -5 & -2 & 1 \\ -3 & y-3 & 3 \end{vmatrix} = (-3-y)\,\vec{\imath} + 12\,\vec{\jmath} + (9-5y)\,\vec{k} = (-3-y, 12,9-5y)$$

On the other hand, since the plane β is perpendicular to the plane OXY, the normal vector of β is $(0,0,1)$. So we obtain that:

$$(-3-y, 12,9-5y)\cdot(0,0,1) = 0 \Rightarrow 9-5y = 0 \Rightarrow y = \frac{9}{5} \Rightarrow C = (3,9/5,4)\ ;\ \vec{n} = (-24/5,12,0)$$

We determine the equation of the plane β using a point, for example the point *A*, and the normal vector obtained:

$$\beta:\ -\frac{24}{5}(x-6) + 12(y-3) + 0(z-1) = 0\ \Rightarrow \beta:\ -4x + 10y - 6 = 0$$

- Compute the plane α:

Since the plane α is parallel to the plane XOZ, both planes have the same normal vector and:

$$\alpha: 0(x-1) + 1(y-1) + 0(z-2) = 0 \Rightarrow \alpha: y = 1$$

- Obtain the intersection between α and β:

$\alpha \cap \beta = \begin{cases} y-1=0 \\ -4x+10y-6=0 \end{cases}$ being $M = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 10 & 0 \end{pmatrix}$ and $M' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -4 & 10 & 0 & 6 \end{pmatrix}$

$rank(M) = 2 = rank(M') <$ number of unknowns, so the planes intersect in a line. The solution of the system is $x = 1; y = 1\ \forall z$. Hence, the intersection line is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} z$$

EXERCISE 4

Find the parallel plane to the line r: $\begin{cases} x+3z=11 \\ y+3z=6 \end{cases}$ *which contains the line* t: $\frac{x-2}{3}=1-y=\frac{z}{3}$.

Solution:

We compute the parametric equations of the line t:

$$\begin{cases} \dfrac{x-2}{3}=1-y \\ \dfrac{x-2}{3}=\dfrac{z}{3} \end{cases} \Rightarrow \begin{cases} x-2=3-3y \\ x-2=z \end{cases} \Rightarrow \begin{cases} x+3y-5=0 \\ x-z-2=0 \end{cases}$$

The sheaf of intersecting planes with common line t, that is to say, the sheaf of planes with edge t is:

$$x+3y-5+\lambda(x-z-2)=0 \Rightarrow (1+\lambda)x+3y-\lambda z-5-2\lambda=0$$

We obtain the parametric equations of the line r doing $z=\lambda$:

r: $\begin{cases} x=11-3\lambda \\ y=6-3\lambda \\ z=\lambda \end{cases}$,so, the direction vector of the line r is $\vec{v}_r=(-3,-3,1)$.

Since, the plane we are looking for is parallel to the line r, the normal vector of the plane is perpendicular to the direction vector of the line:

$$\vec{n}_\pi\cdot\vec{v}_r=0 \Rightarrow (1+\lambda,3,-\lambda)\cdot(-3,-3,1)=0 \Rightarrow \lambda=-3$$

Substituting this value in the equation of the sheaf of intersecting planes computed above, we obtain the equation of the plane π:

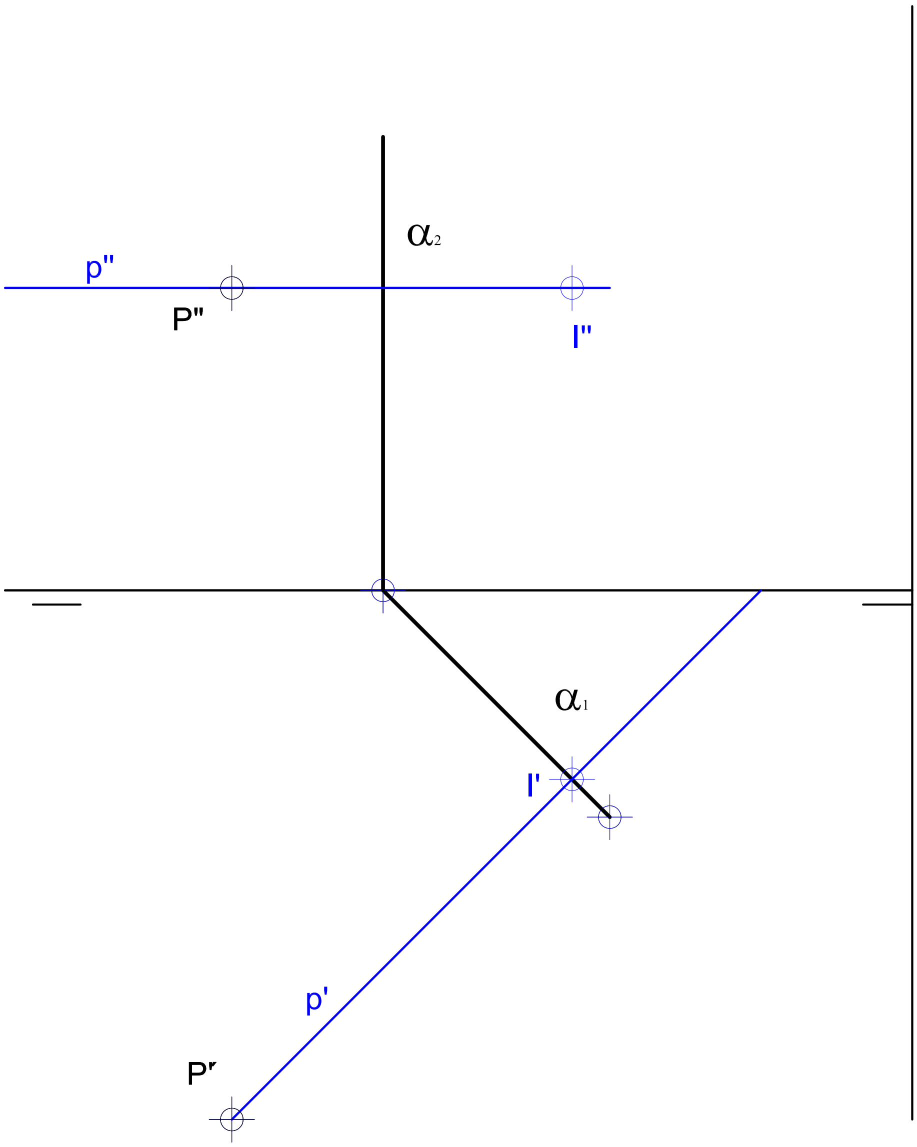
$$(1+(-3))x+3y-(-3)z-5-2(-3)=0 \Rightarrow \pi: -2x+3y+3z+1=0$$

EXERCISE 5

Find the line that passing through the point $P(9,7,4)$*, is perpendicular to the plane* α*. This plane contains the points* $(7,0,0)$ *and* $(4,3,0)$*, and it is perpendicular to the plane* $z=0$*. Calculate the point of intersection between them*

Draw the line p *that passing through the point* P *is perpendicular to the plane* α*.*

Calculate the point of intersection I *between them.*



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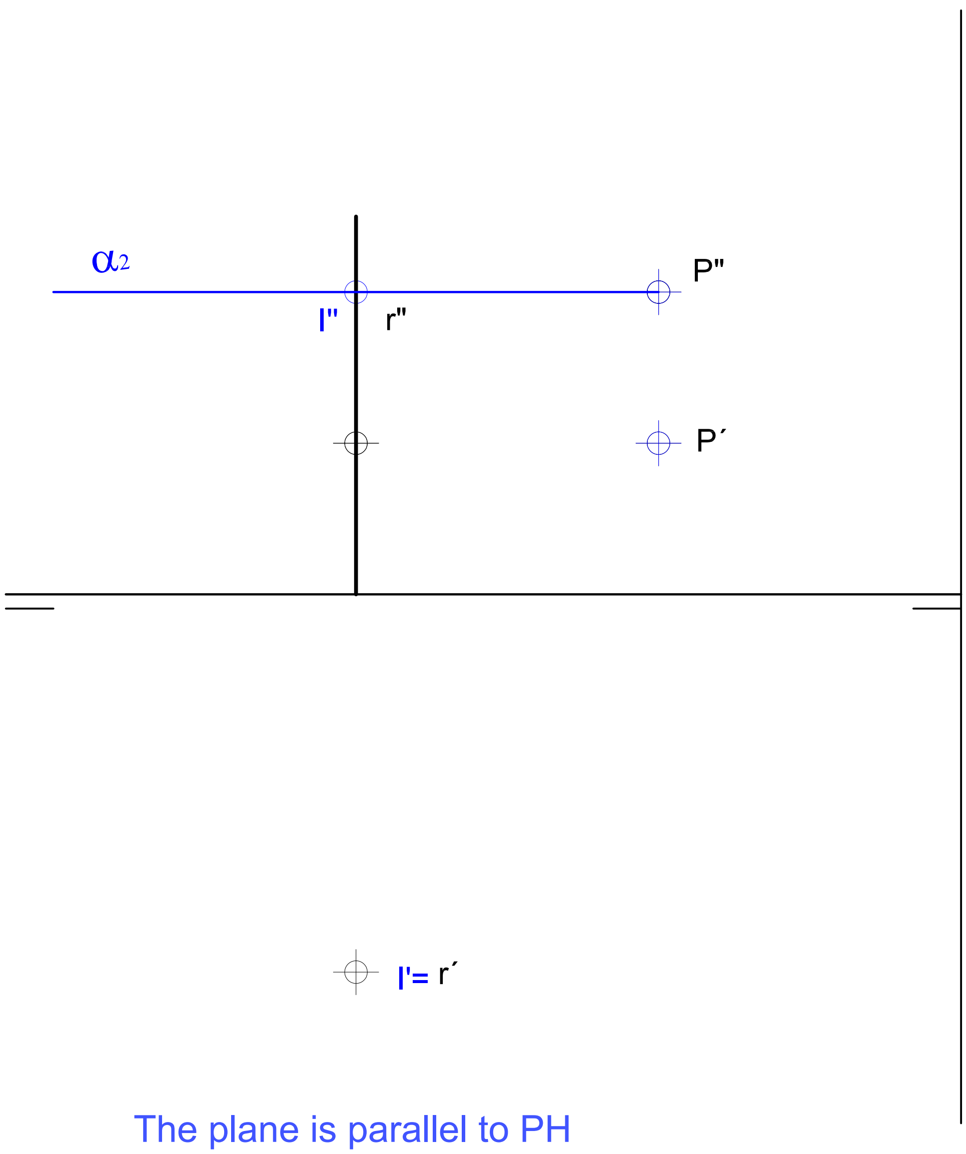
EXERCISE 6

Find the plane that passing through the point $P(2,-2,4)$ *is perpendicular to the line that passing through the point* $(8,5,2)$ *is perpendicular to the plane XOY. Calculate the point of intersection between them.*

Draw the plane α that contains the point P and is perpendicuar to the line r. Find the intersection between them (I).



The plane is parallel to PH

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EXERCISE 7

Draw the line that passing through the point P(12,3,6) is perpendicular to the line r that passes through (6,5,4) and (0,8,7), and is parallel to the plane α determined by the points (11,0,0), (6,0,2) and (8,4,0).

Solution:

The procedure to compute the requested line is the following:

- Obtain the line r :

  Let denote $A=(6,5,4)$ and $B=(0,8,7)$, then the direction vector of the line r is $\overrightarrow{AB}=(-6,3,3)$. Hence, the continuous equation of the line r is:

$$r: \frac{x}{-6}=\frac{y-8}{3}=\frac{z-7}{3}$$

- Obtain the plane α:

  Let denote $A=(11,0,0), B=(6,0,2)$ and $C=(8,4,0)$, then $\overrightarrow{AB}=(-5,0,2)$, $\overrightarrow{AC}=(-3,4,0)$. Therefore, the implicit equation of the plane α is:

$$\alpha: \begin{vmatrix} x-11 & -5 & -3 \\ y & 0 & 4 \\ z & 2 & 0 \end{vmatrix} = 4x+3y+10z-44=0$$

- Compute the requested line s:

  The direction vector of the line we are looking for is perpendicular to the direction vector of the line r and to the normal vector of the plane α:

$$\vec{v}_s=\vec{v}_r \wedge \vec{n}_\alpha = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 1 & 1 \\ 4 & 3 & 10 \end{vmatrix} = 7\,\vec{\imath}+24\,\vec{\jmath}-10\vec{k}=(7,24,-10)$$

  Then, the equation of the line s is:

$$s: \frac{x-12}{7}=\frac{y-3}{24}=\frac{z-6}{-10}$$

EXERCISE 10

Calculate the distance between the points $A(4,8,6)$ *and* $B(4,3,3)$.

Solution:

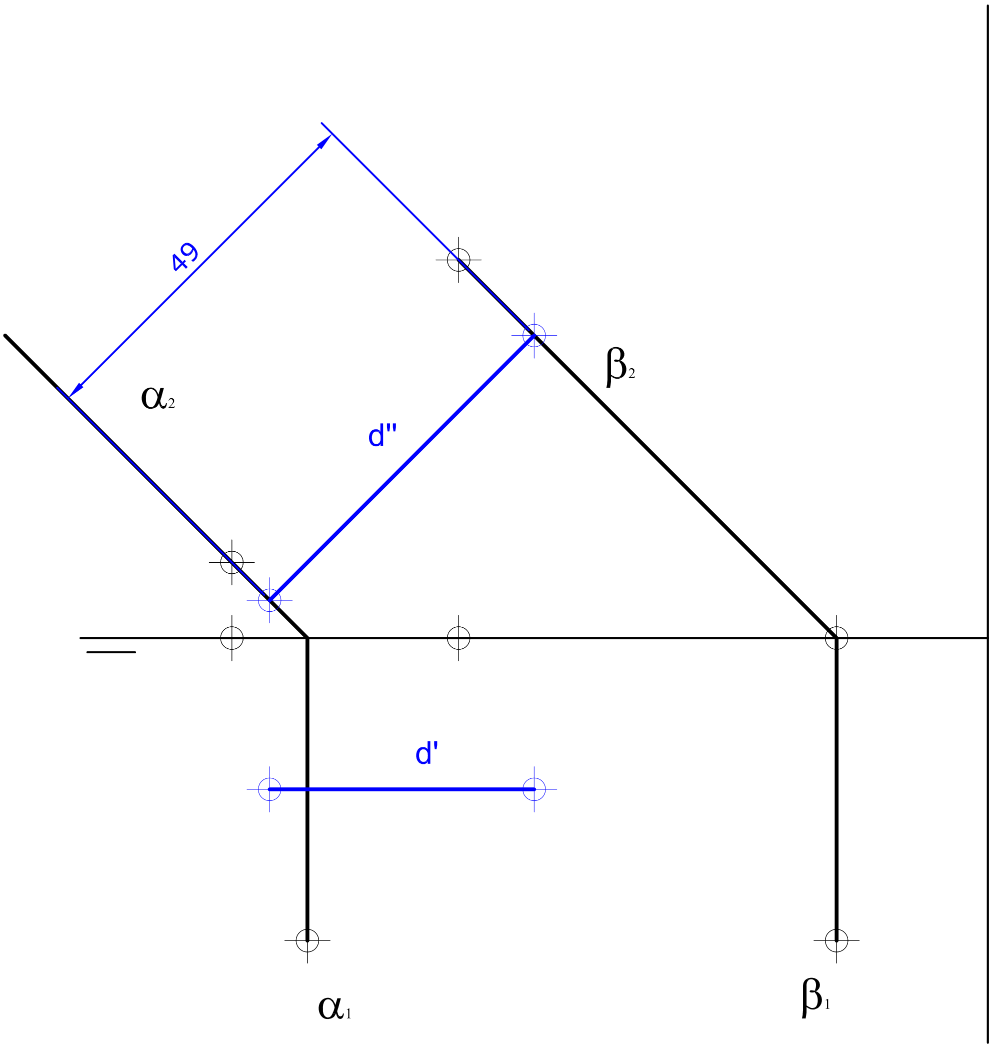
The distance between two points can be calculated using the next expression:

$$d(A,B)=\sqrt{(4-4)^2+(3-8)^2+(3-6)^2}=\sqrt{25+9}=\sqrt{34}$$

EXERCISE 13

Let α be a plane determined by the points, $(9,0,0)$, $(10,0,1)$ and $(9,4,0)$, and β determined by $(2,0,0)$, $(7,0,5)$ and $(2,4,0)$. Calculate the distance between these planes.

Calculate the distance between the planes α and β .



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EXERCISE 14

Let α be a plane determined by the points $(9,0,0)$, $(10,0,1)$ and $(9,4,0)$, and the plane β determined by $(2,0,0)$, $(7,0,5)$ and $(2,4,0)$. Calculate the bisector plane of α and β.

<u>Solution:</u>

The procedure to obtain the bisector plane of α and β is the following:

- Calculate the plane α:

  Let denote the points $A=(9,0,0)$, $B=(10,0,1)$ and $C=(9,4,0)$. We obtain the normal vector of the plane using the vectors $\overrightarrow{AB}=(1,0,1)$ and $\overrightarrow{AC}=(0,4,0)$:

$$\vec{n}_\alpha = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 9 & 0 & 0 \\ 0 & 4 & 0 \end{vmatrix} = -4\,\vec{\imath} + 0\,\vec{\jmath} + 4\vec{k} = (-4,0,4)$$

  Then, the equation of the plane α that passing through the point A has the associated vector $\vec{n}_\alpha$ is:

$$-4(x-9)+0(y-0)+4(z-0) \Rightarrow \ \alpha\colon x-z-9=0$$

- Obtain the plane β:

  Let denote $D=(2,0,0)$, $E=(7,0,5)$ and $F=(2,4,0)$. In the same way, we obtain the normal vector of the plane β using the vectors $\overrightarrow{DE}=(5,0,5)$ and $\overrightarrow{DF}=(0,4,0)$:

$$\vec{n}_\beta = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 5 & 0 & 5 \\ 0 & 4 & 0 \end{vmatrix} = -20\,\vec{\imath} + 0\,\vec{\jmath} + 20\vec{k} = (-20,0,20)$$

  Then, the equation of the plane β that passing through the point D has the associated vector $\vec{n}_\beta$ is:

$$-20(x-2)+0(y-0)+20(z-0) \Rightarrow \ \beta\colon x-z-2=0$$

- Obtain the bisector plane:

  First, we compute the line that passing through the point D is perpendicular to the planes α and β:

$$r\colon \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix} \Rightarrow \begin{cases} x = 2-4\lambda \\ y=0 \\ z=4\lambda \end{cases}$$

  Next, we calculate the intersection between the line r and the plane α:

$$(2-4\lambda)-4\lambda-9=0 \Rightarrow \lambda=-\frac{7}{8} \Rightarrow \begin{cases} x=2+7/2 \\ y=0 \\ z=-7/2 \end{cases} \Rightarrow P = r\cap\alpha = \left(\frac{11}{2},0,-\frac{7}{2}\right)$$

And we obtain the midpoint of $\overline{DP}$:

$$M = \frac{D+P}{2} = \frac{(2,0,0) + \left(\frac{11}{2}, 0, -\frac{7}{2}\right)}{2} = \left(\frac{15}{4}, 0, -\frac{7}{4}\right)$$

Finally, since the bisector plane contains the point M, this point satisfies the equation of the bisector plane, $x - z + K = 0$:

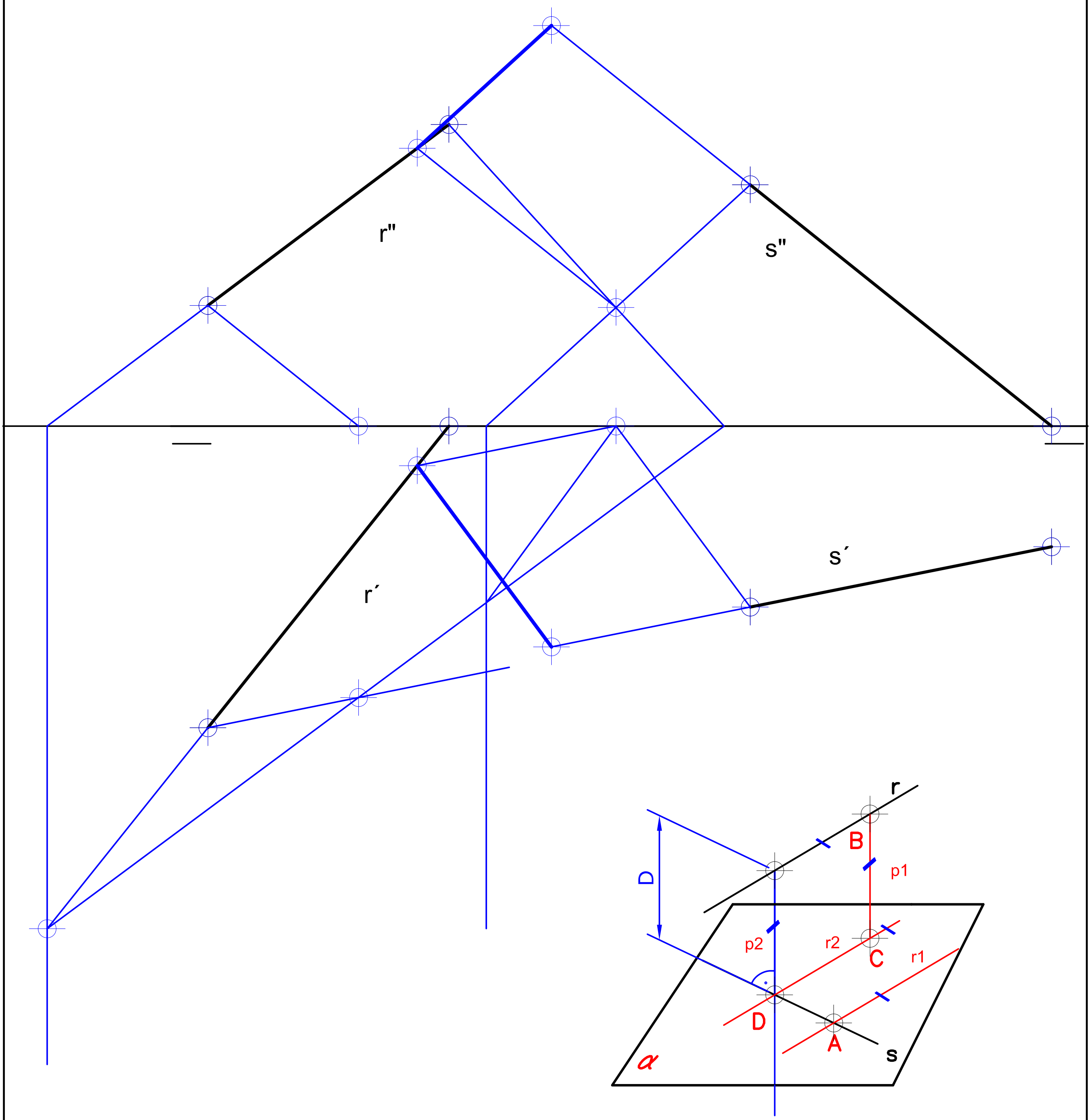
$$\frac{15}{4} - \left(-\frac{7}{4}\right) + K = 0 \Rightarrow K = -\frac{22}{4}$$

Hence, the bisector plane we are looking for is $x - z - \frac{22}{4} = 0$.

EXERCISE 18

Calculate the distance between the lines $r\big((13,0,5)(17,5,2)\big)$ *and* $s\big((3,2,0)(8,3,4)\big)$.

Calculate the distance between the lines r and s.



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EXERCISE 19

Calculate the distance from the point $A(1,2,5)$ *to the plane* $\alpha : 2x+2y-z-5=0$.

Solution:

First of all, we verify that the plane α does not contain the point A:

$$2\cdot 1+2\cdot 2-5-5=-4\neq 0 \Rightarrow A\notin\alpha$$

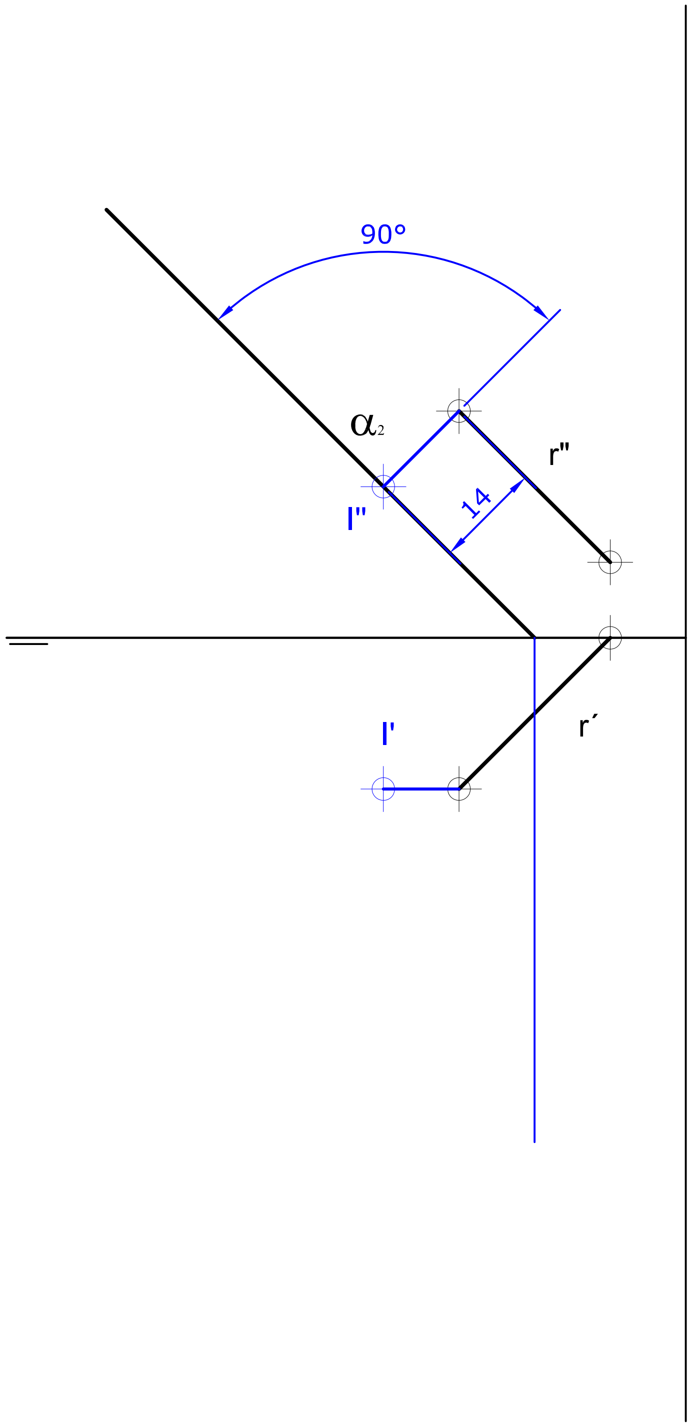
The distance between a point and a plane can be calculated using the next expression:

$$d(A,\alpha)=\frac{|2\cdot 1+2\cdot 2+(-1)\cdot 5-5|}{\sqrt{1^2+2^2+5^2}}=\frac{|-4|}{\sqrt{30}}=\frac{4}{\sqrt{30}}$$

EXERCISE 20

Calculate the distance from the line $r: \dfrac{x-1}{2} = \dfrac{y}{2} = \dfrac{z-1}{2}$ *to the plane* $\alpha : x - z = 2$.

Calculate the distance between the plane α (perpendicular to PV) and the line r.



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EXERCISE 21

Calculate the distance from the point $P(1,3,-1)$ *to the line* $r:\begin{cases} x-y=0 \\ x+y-z=0 \end{cases}$.

Solution:

First of all, we verify that the point P is not included in the line r: $\begin{cases} 1-3\neq 0 \\ 1+3-(-1)\neq 0 \end{cases} \Rightarrow P\notin r$

Next, we obtain the parametric equations of the line using its implicit equations:

$$\begin{cases} x-y=0 \\ x+y-z=0 \end{cases} \Rightarrow \begin{cases} x-y=0 \\ x+y=z \end{cases} \Rightarrow x=\frac{z}{2}, y=\frac{z}{2}$$

That is: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix}$, so, $\vec{v}_r=(1,1,2)$.

Finally, using any point of the line r, for example the point $A=(0,0,0)$, building the vector $\overrightarrow{AP}=(1,3,-1)$ and applying the expression to calculate the distance between a point and a line, we obtain that:

$$d(P,r)=\frac{\left|\overrightarrow{AP}\wedge\vec{v}_r\right|}{|\vec{v}_r|}=\frac{\sqrt{7^2+(-3)^2+(-2)^2}}{\sqrt{1^2+1^2+2^2}}=\sqrt{\frac{62}{6}}=\sqrt{\frac{31}{3}}$$

since $\overrightarrow{AP}\wedge\vec{v}_r=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & -1 \\ 1 & 1 & 2 \end{vmatrix}=7\,\vec{i}-3\,\vec{j}-2\vec{k}$.

EXERCISE 22

Calculate the distance between the lines $r: \frac{x+3}{3}=\frac{y-9}{-2}=\frac{z-8}{-2}$ *and* $s: \frac{x-3}{-2}=\frac{y-2}{1}=\frac{z-1}{2}$.

Solution:

We consider the vectors $\vec{v}_r = (3,-2,-2)$ and $\vec{v}_s = (-2,1,2)$, direction vectors of each line respectively. These two vectors are not parallel, hence the two lines are skew or the lines intersect in one point.

To determine if the lines are skew or they intersect, we consider a point of each line $A = (-3,9,8) \in r$ and $B = (6,-7,-7) \in s$, we build the vector $\overrightarrow{AB} = (6,-7,-7)$ and we study the rank of the matrix $(\vec{v}_r, \vec{v}_s, \overrightarrow{AB})$:

$$rank(\vec{v}_r, \vec{v}_s, \overrightarrow{AB}) = rank\begin{pmatrix} 3 & -2 & 6 \\ -2 & 1 & -7 \\ -2 & 2 & -7 \end{pmatrix} = 3$$

Then, $rank(\vec{v}_r, \vec{v}_s) = 2 \neq rank(\vec{v}_r, \vec{v}_s, \overrightarrow{AB}) = 3$, so, the lines are skew. The distance between two skew lines is given by the following expression:

$$d(r,s) = \frac{|(\overrightarrow{AB}, \vec{v}_r, \vec{v}_s)|}{|\vec{v}_r \wedge \vec{v}_s|} = \frac{|\overrightarrow{AB} \cdot (\vec{v}_r \wedge \vec{v}_s)|}{|\vec{v}_r \wedge \vec{v}_s|} = \frac{|(6,-7,-7) \cdot (-2,-2,-1)|}{3} = \frac{9}{3} = 3$$

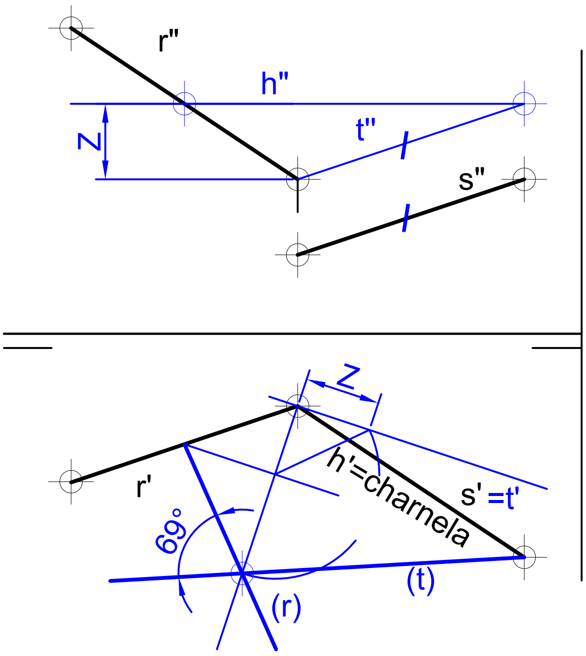
since

$$\vec{v}_r \wedge \vec{v}_s = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3 & -2 & -2 \\ -2 & 1 & 2 \end{vmatrix} = -2\,\vec{\imath} - 2\,\vec{\jmath} - \vec{k} \Rightarrow |\vec{v}_r \wedge \vec{v}_s| = \sqrt{(-2)^2 + (-2)^2 + (-1)^2} = 3$$

EXERCISE 23

Calculate the angle between the line $r:\begin{cases} x-3y=1 \\ 2y=z \end{cases}$ *and the line* s *that passes through the points* $(4,1,1)$ *and* $(1,3,3)$.

Find the angle between the lines r and s.



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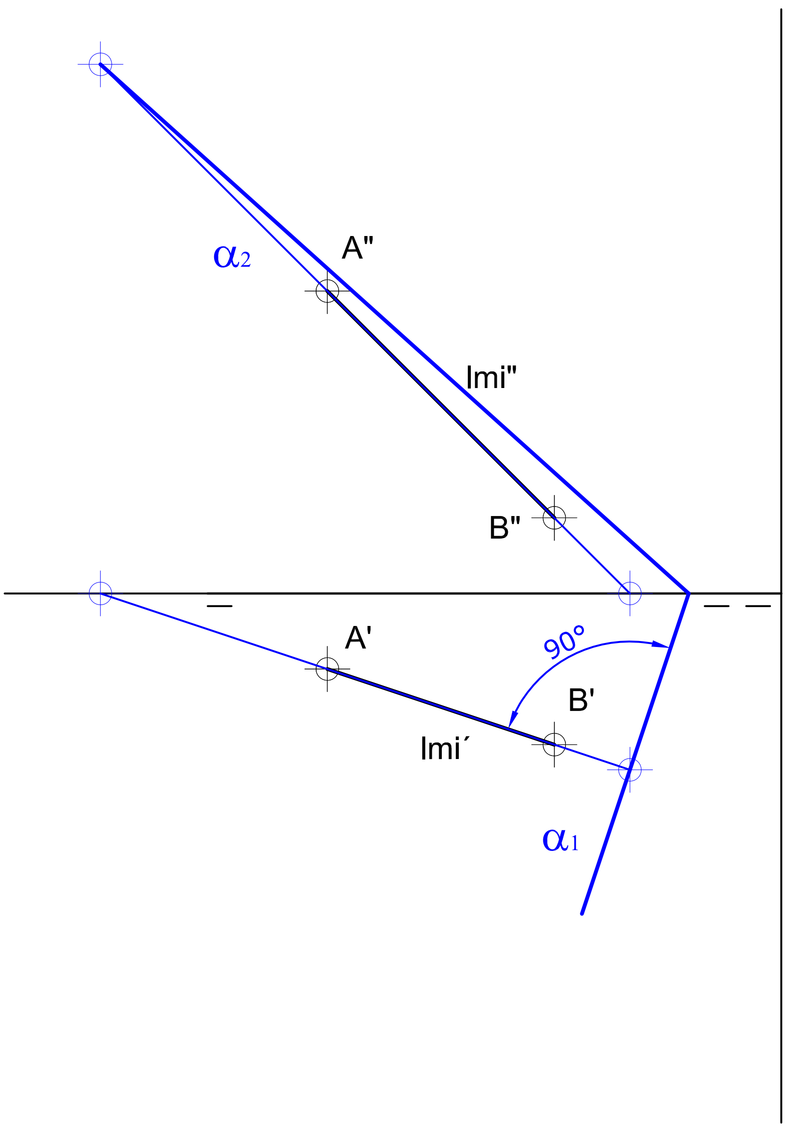
EXERCISE 27

Define the plane α *, being the line* $s:\begin{cases} x+3y=9 \\ 3y+z=7 \end{cases}$ *its line of maximum inclination.*

Find the plane α , being lmi its line of maximum inclination.



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EXERCISE 30

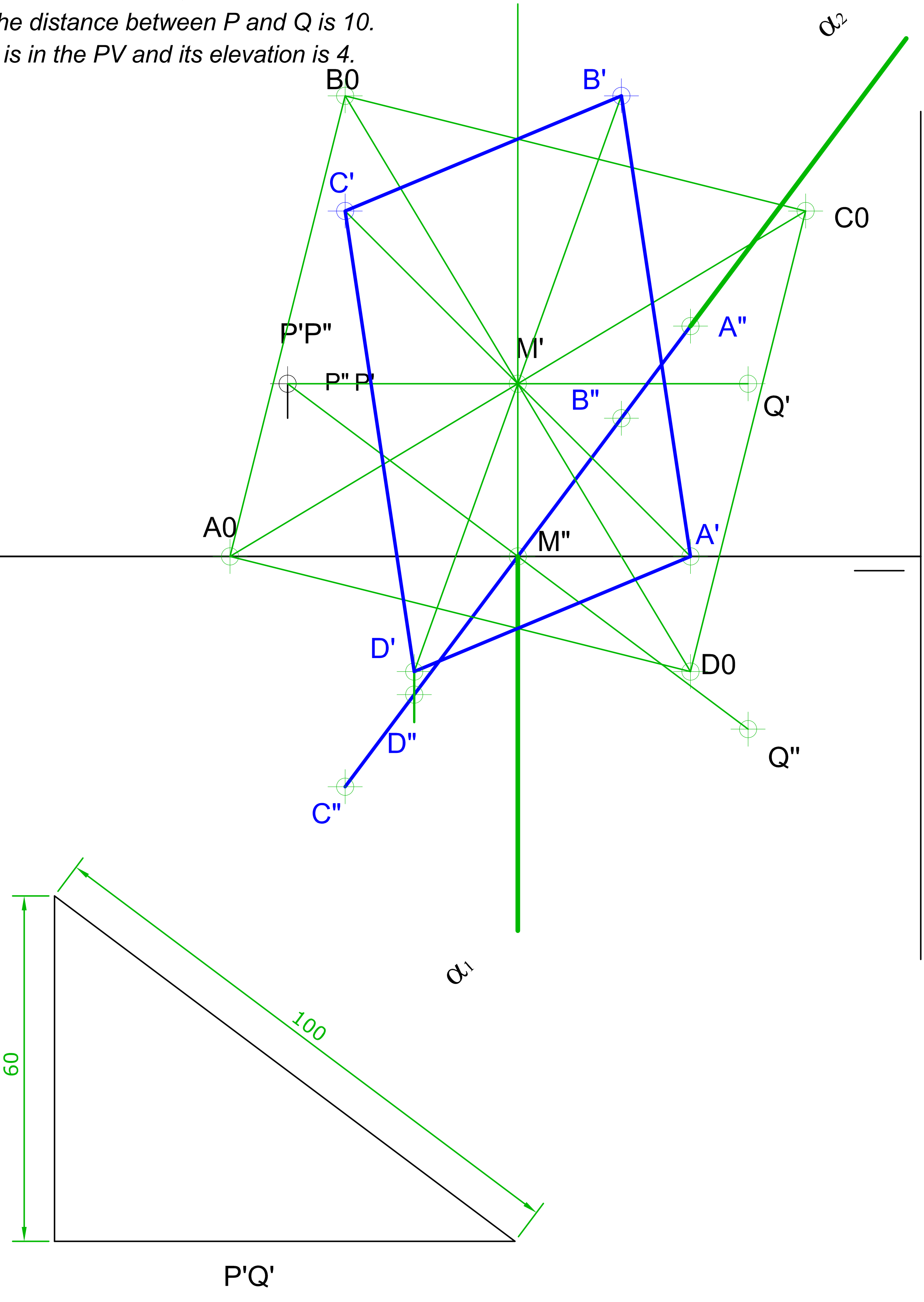
Let $P(11,-3,3)$ and $Q(--,-3,-3)$ be two points. Define the vertex of a square ABCD knowing that:

- *The vertex of the square are equidistant from P and Q.*
- *The distance between the points P and Q is 10.*
- *The point A is included in the plane $y=0$.*
- *The third coordinate of the point A (its height) is 4.*

Draw the square of vertexes ABCD equidistant to the points P and Q.

Data:

1. *The elevation of Q is -3 and it is in the first bisector.*
2. *The distance between P and Q is 10.*
3. *A is in the PV and its elevation is 4.*



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