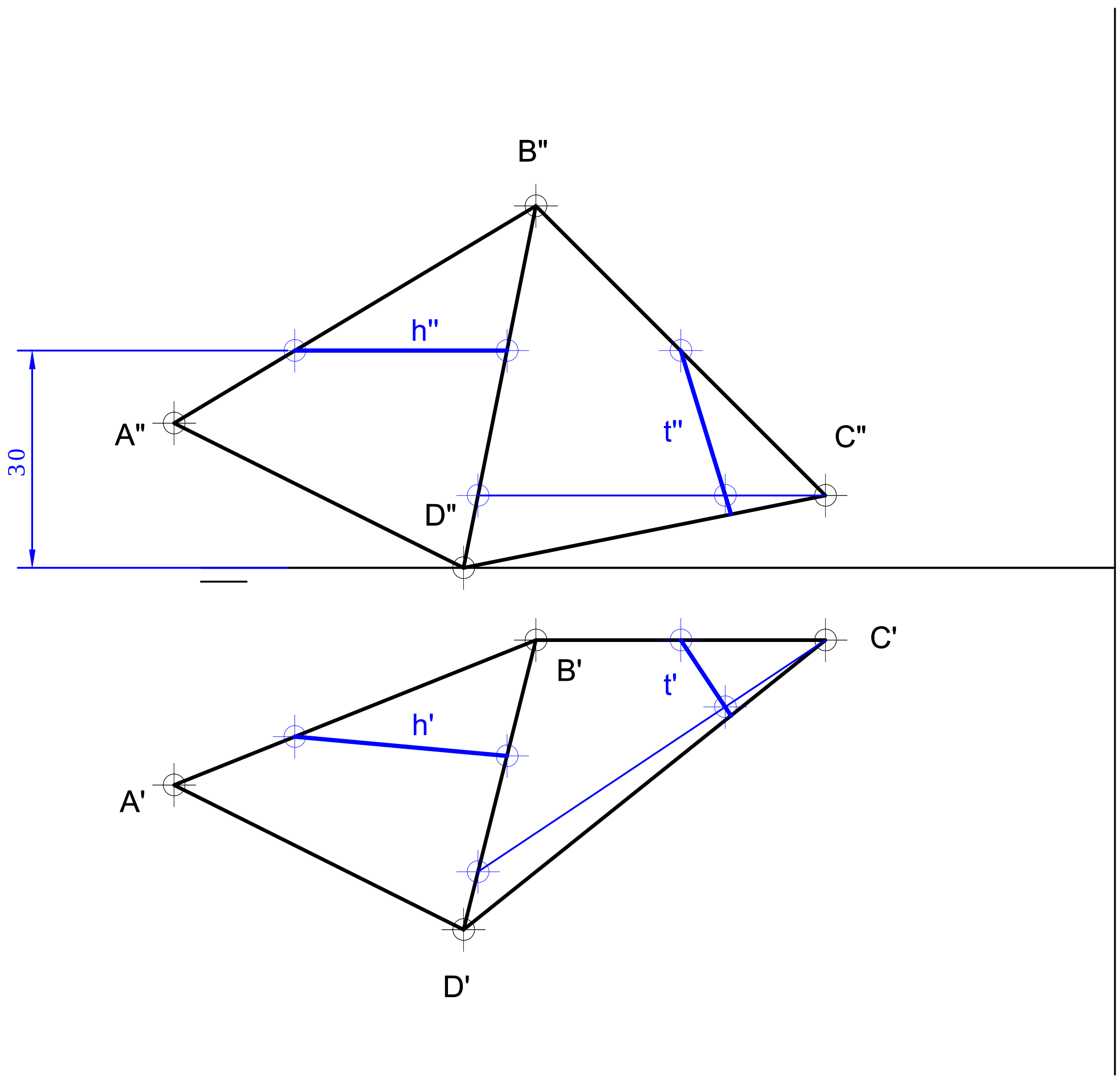
EXERCISE 1

Let $A(13,3,2)$, $B(8,1,5)$, $C(4,1,1)$ *and* $D(9,5,0)$ *be four points, being ABC and BDC two planes that define a part of a roof.*

- *Find a line in the plane ABD, parallel to the plane XOY and being the height of the points of this line 3 (z=3).*
- *Define the trajectory of a drop that leaves from the midpoint of the segment BC.*

ABD and BDC are two planes that define a roof. Draw a horizontal line with an elevation of 3 that is located in the plane ABD. Define the trajectory of a drop that leaves from de midpoint of the segment BC.



The line "t" is the trajectory of the drop.

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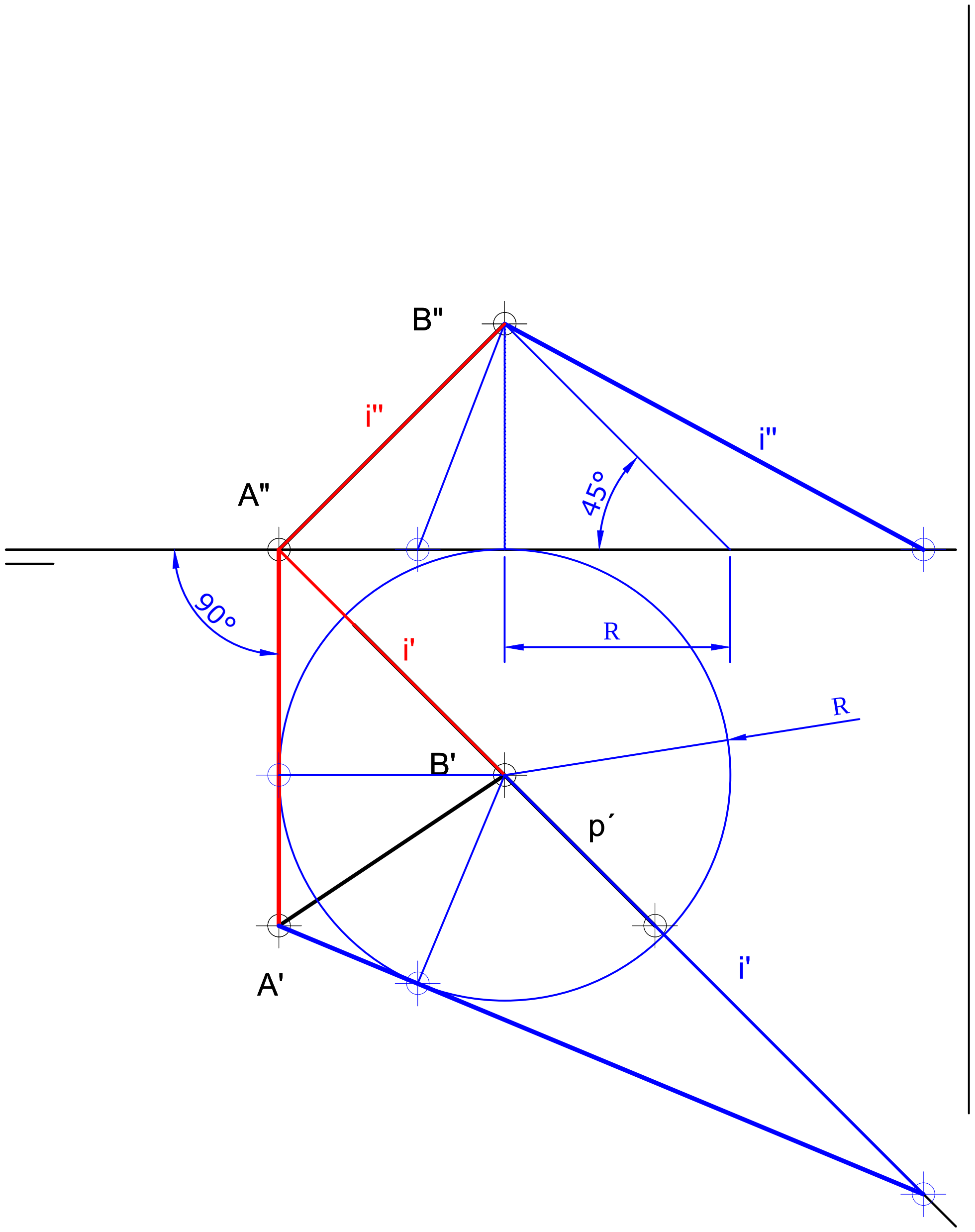
EXERCISE 2

Let $A(9,5,0)$ *and* $B(6,3,3)$ *be two points included in the line of intersection of two symmetric sheets of concrete.*

- *Define the previous planes being their slope 45º.*
- *Calculate the intersection of these two planes with a vertical plane that contains the line that passes through the points* $(6,3,z)$ *and* $(4,5,z)$, *and the intersection with the horizontal plane.*

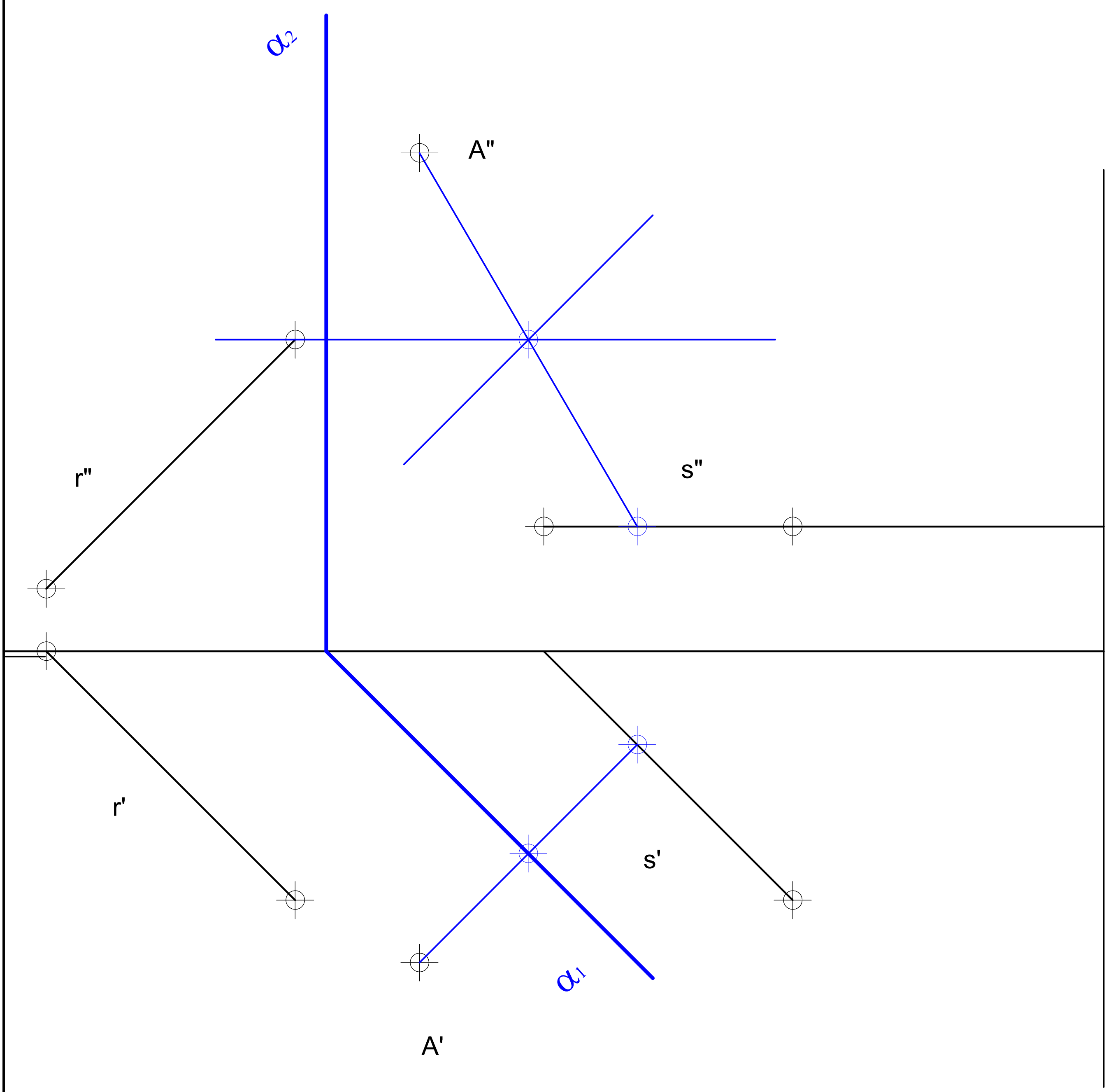
AB is the intersection of two symmetric sheets of concrete. Draw these two planes and their intersections with a vertical plane that contains the line "p", and the intersection with the horizontal plane.
Data: the slope of the planes = 45º.



EXERCISE 3

Let s be a line that passes through the points $B(9,0,2)$ and $C(5,4,2)$, and r a line that passes through $D(13,4,5)$ and $E(17,0,1)$. Find the plane that is equidistant to the point $A(11,5,8)$ and to the line s, and parallel to the line r.

Draw a plane that is equidistant to the point A and to the line s, and parallel to the line r.



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EXERCISE 1

Let $A(13,3,2)$, $B(8,1,5)$, $C(4,1,1)$ *and* $D(9,5,0)$ *be four points, being ABC and BDC two planes that define a part of a roof.*

- *Find a line in the plane ABD, parallel to the plane XOY and being the height of the points of this line 3 (z=3).*
- *Define the trajectory of a drop that leaves from the midpoint of the segment BC.*

<u>*Solution:*</u>

First of all we will calculate the implicit equation of the plane ABD. We will use the point $A(13,3,2)$ and the normal vector $\vec{n}_{ABC}$. This vector can be calculated by doing the vector product of $\overrightarrow{AB}=(-5,-2,3)$ and $\overrightarrow{AD}=(-4,2,-2)$.

$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -5 & -2 & 3 \\ -4 & 2 & -2 \end{vmatrix} = -2\vec{\imath}-22\vec{\jmath}-18\vec{k} \Rightarrow \vec{n}_{ABC}=(1,11,9)$$

Hence, the implicit equation of the plane ABD is this one:

$$(x-13)+11(y-3)+9(z-2)=0 \Rightarrow$$

$$ABD\colon x+11y+9z-64=0$$

We need the point $P=(x,y,3)$ included in the line and the direction vector of this line $\vec{v}_r=(a,b,c)$ to calculate the line included in the plane ABD.

As the line r is included in the plane, its direction vector $\vec{v}_r$ is perpendicular to the vector, $\vec{n}_{ABC}$. As the line is parallel to the plane XOY, the vector $\vec{v}_r$ is perpendicular to the vector $(0,0,1)$. Therefore:

$$\begin{cases} \vec{v}_r \perp \vec{n}_{ABC} \Rightarrow (a,b,c)\cdot(1,11,9)=0 \\ \vec{v}_r \perp (0,0,1) \Rightarrow (a,b,c)\cdot(0,0,1)=0 \end{cases} \Rightarrow \begin{cases} a=-11b \\ c=0 \end{cases} \Rightarrow \vec{v}_r=(-11b,b,0)$$

We will consider $\vec{v}_r=(-11,1,0)$ as the direction vector. We only have to determine the point $P=(x,y,3)$ and to do this, it is enough to take into account that the point is included in the line. Furthermore, as the line is included in the plane ABD, the point is also in the plane. This means that the x and y coordinates of the point have to satisfy:

$$x+11y=64-27 \Rightarrow x+11y=37$$

For example, taking $x=4$ and $y=3$, the point is $P=(4,3,3)$.

Hence, the direction vector of the searched line is $\vec{v}_r = (-11,1,0)$ and the point $P = (4,3,3)$ is included in the line. The parametric equations of the line are the following:

$$\begin{cases} x = 44 - 11t \\ y = 3 + t \\ z = 3 \end{cases}$$

And the implicit equations of the line are:

$$\begin{cases} x + 11y - 37 = 0 \\ z = 3 \end{cases}$$

- *Define the trajectory of a drop that leaves from the midpoint of the segment BC.*

We start by calculating the midpoint of the segment BC:

$$M = \frac{B+C}{2} = (6,1,3)$$

The drop will follow the trajectory of the line of maximum slope. We have to follow the next procedure:

- Calculate the plane BDC that contains the line BC. Obtain the line of intersection r between this plane and the plane OXY.

We will consider the vectors $\overrightarrow{BD} = (1,4,-5)$ and $\overrightarrow{BC} = (-4,0,-4)$ to calculate the plane BDC. We will use the point $B(8,1,5)$ to determine the plane. Therefore, the implicit equation of plane BDC is:

$$\begin{vmatrix} x-8 & 1 & -4 \\ y-1 & 4 & 0 \\ z-5 & -5 & -4 \end{vmatrix} = 0 \Rightarrow BDC: \quad -2x + 3y + 2z + 3 = 0$$

The line r is the intersection between the planes BDC and OXY, being its implicit equations:

$$r: \begin{cases} 2x + 3y + 2z + 3 = 0 \\ z = 0 \end{cases}$$

- We calculate the plane π, that contains the point M and is perpendicular to r:

As the plane is perpendicular to the line, the direction vector of the line $\vec{v}_r$ and the normal vector of the plane $\vec{n}_\pi$ are parallel. We can consider that these two vectors are the same, taking this value:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 3\vec{i} + 2\vec{j} \Rightarrow \vec{n}_\pi = \vec{v}_r = (3,2,0)$$

Therefore, the plane π is given by:

$$\pi: 3x + 2y - 20 = 0$$

- We calculate the line of maximum slope.

This line is the intersection between the planes π and BDC:

$$\begin{cases} 3x + 2y - 20 = 0 \\ -2x + 3y + 2z + 3 = 0 \end{cases}$$

EXERCISE 2

Let $A(9,5,0)$ and $B(6,3,3)$ be two points included in the line of intersection of two symmetric sheets of concrete.

- *Define the previous planes being their slope 45º.*
- *Calculate the intersection of these two planes with a vertical plane that contains the line that passes through the points $(6,3,z)$ and $(4,5,z)$, and the intersection with the horizontal plane.*

<u>Solution:</u>

We start the exercise by calculating the line r that passes through the points $A(9,5,0)$ and $B(6,3,3)$. The direction vector of this line is: $\overrightarrow{AB} = B - A = (6,3,3) - (9,5,0) = (-3,-2,3)$. Hence, the line r is given by this expression:

$$r: \frac{x-9}{-3} = \frac{y-5}{-2} = \frac{z-0}{3}$$

We calculate two planes that contain the line r:

$$r: \begin{cases} \dfrac{x-9}{-3} = \dfrac{y-5}{-2} \\ \dfrac{x-9}{-3} = \dfrac{z}{3} \end{cases} \Rightarrow \begin{cases} 2x - 3y - 33 = 0 \\ x + z - 9 = 0 \end{cases}$$

The sheaf of planes that contains the line r is:

$$(2x - 3y - 33) + \lambda(x + z - 9) = 0, \qquad \lambda \in \mathbb{R} \Rightarrow$$

$$\pi: (2+\lambda)x + 3y - \lambda z - 33 - 9\lambda = 0, \qquad \lambda \in \mathbb{R}$$

The normal vector of the planes of the sheaf of planes is $\vec{n}_\pi = (2+\lambda, 3, -\lambda)$. This is also the normal vector of the planes α and β that we are looking for. These planes form an angle of 45º with the horizontal plane (plane $z = 0$). The normal vector of this plane is $\vec{n}_{OXY} = (0,0,1)$. By applying the formula to calculate the angle between two planes, we get:

$$\theta = arccos\left(\frac{|\vec{n}_{OXY} \cdot \vec{n}_\pi|}{|\vec{n}_{OXY}||\vec{n}_\pi|}\right) \Rightarrow cos\theta = \frac{|\vec{n}_{OXY} \cdot \vec{n}_\pi|}{|\vec{n}_{OXY}||\vec{n}_\pi|} \Rightarrow cos45^\circ = \frac{|-\lambda|}{\sqrt{(2+\lambda)^2 + \lambda^2 + 3^2}}$$

$$\frac{1}{\sqrt{2}} = \frac{\lambda}{\sqrt{(2+\lambda)^2 + \lambda^2 + 3^2}} \Rightarrow \frac{1}{2} = \frac{\lambda^2}{(2+\lambda)^2 + \lambda^2 + 3^2} \Rightarrow (2+\lambda)^2 + \lambda^2 + 3^2 = 2\lambda^2 \Rightarrow \lambda = -\frac{13}{4}$$

One of the planes is this one:

$$\alpha: -\frac{5}{4}x + 3y + \frac{13}{4}z - \frac{15}{4} = 0$$

Taking into account that the planes are perpendicular and their normal vector is $\vec{n}_\pi = (2+\lambda, 3, -\lambda)$, the normal vector of the plane β can be calculated:

$$\vec{n}_\alpha \perp \vec{n}_\beta \Rightarrow= \left(-\frac{5}{4}, 3, \frac{13}{4}\right) \cdot (2+\lambda, 3, -\lambda) = 0 \Rightarrow \frac{9}{2}\lambda = \frac{13}{2} \Rightarrow \lambda = \frac{13}{9}$$

And β is given by:

$$\beta: \frac{31}{9}x + 3y - \frac{13}{9}z - \frac{15}{4} = 0$$

Finally, the intersection between the planes α and β with the line that passes through the points $P(6,3,a)$ and $Q(4,5,a)$ is calculated. We define the line s that contains these two points.

$\overrightarrow{PQ} = Q - P = (4,5,a) - (6,3,a) = (-2,2,0)$. And we write the parametric equations of the line s:

$$s: \begin{cases} x = 4 - 2t \\ y = 5 + 2t \\ z = a \end{cases}$$

We calculate the point of intersection of s with the plane α:

$$-\frac{5}{4}(4-2t) + 3(5+2t) + \frac{13}{4}a - \frac{15}{4} = 0 \Rightarrow 34t = -25 - 13a \Rightarrow t = \frac{-25-13a}{34}$$

Hence, the point of intersection is: $S_1\left(\frac{93+13a}{17}, \frac{60-13a}{17}, a\right)$.

In the same way, we calculate the point of intersection of s with the plane β:

$$\frac{31}{9}(4-2t) + 3(5+2t) - \frac{13}{9}a - \frac{15}{4} = 0 \Rightarrow -8t = -\frac{901}{4} + 13a \Rightarrow t = \frac{901-52a}{32}$$

Being the point of intersection: $S_2\left(\frac{-837+52a}{16}, \frac{981-52a}{16}, a\right)$.

EXERCISE 3

Let s be a line that passes through the points $B(9,0,2)$ and $C(5,4,2)$, and r a line that passes through $D(13,4,5)$ and $E(17,0,1)$. Find the plane that is equidistant to the point $A(11,5,8)$ and to the line s, and parallel to the line r.

Solution:

First, we calculate the parametric equations of the lines s and r:

$$\overrightarrow{BC} = (-4,4,0) \Rightarrow s\colon \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\overrightarrow{DE} = (4,-4,-4) \Rightarrow r\colon \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 17 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Next, we calculate the plane π that contains the point A and is perpendicular to the line s. As the plane π and the line s are perpendicular, the vectors $\vec{n}_\pi$ and $\vec{v}_s$ are parallel. Therefore, $\vec{n}_\pi = (-1,1,0)$. And π is given by:

$$\pi\colon -1(x-11) + 1(y-5) + 0(z-8) \Rightarrow \pi\colon x - y - 6 = 0$$

We follow the process by calculating the point of intersection P between the plane π and the line s. After having calculated P, the midpoint M of the segment $\overline{AP}$ is calculated:

$$\begin{cases} x - y - 6 = 0 \\ x = 9 - \lambda \\ y = \lambda \\ z = 2 \end{cases} \Rightarrow \lambda = \frac{3}{2} \Rightarrow P = \left(\frac{15}{2}, \frac{3}{2}, 2\right)$$

$$M = \frac{A+P}{2} = \frac{(11,5,8) + \left(\frac{15}{2}, \frac{3}{2}, 2\right)}{2} = \left(\frac{37}{4}, \frac{13}{4}, 5\right)$$

Now we find the plane π', that contains M and is perpendicular to the line s. Let $\vec{v}_{\pi'} = (a,b,c)$ be the normal vector of π'. As this plane and the line s are perpendicular, their scalar product is zero:

$$\vec{v}_{\pi'} \cdot \vec{v}_S = 0 \Rightarrow (a,b,c) \cdot (-1,1,0) = 0 \Rightarrow -a + b = 0 \Rightarrow a = b \ \ \forall c \Rightarrow \vec{v}_{\pi'} = (a,a,c)$$

As π' and r are parallel, the normal vector of the plane and the direction vector of the line are perpendicular (scalar product zero):

$$\vec{v}_{\pi'} \cdot \vec{v}_r = 0 \Rightarrow (a,a,c) \cdot (1,-1,-1) \Rightarrow a - a - c = 0) \Rightarrow c = 0$$

As a consequence, the normal vector of the plane π' is given by $\vec{v}_{\pi'} = (a,a,0)$ da. We will consider $\vec{v}_{\pi'} = (1,1,0)$ as its normal vector.

Finally, the equation of the plane π' can be calculated using its normal vector and the point M included in this plane:

$$\pi'\colon 1\left(x - \frac{37}{4}\right) + 1\left(y - \frac{13}{4}\right) + 0(z-5) \Rightarrow x + y - \frac{50}{4} = 0$$