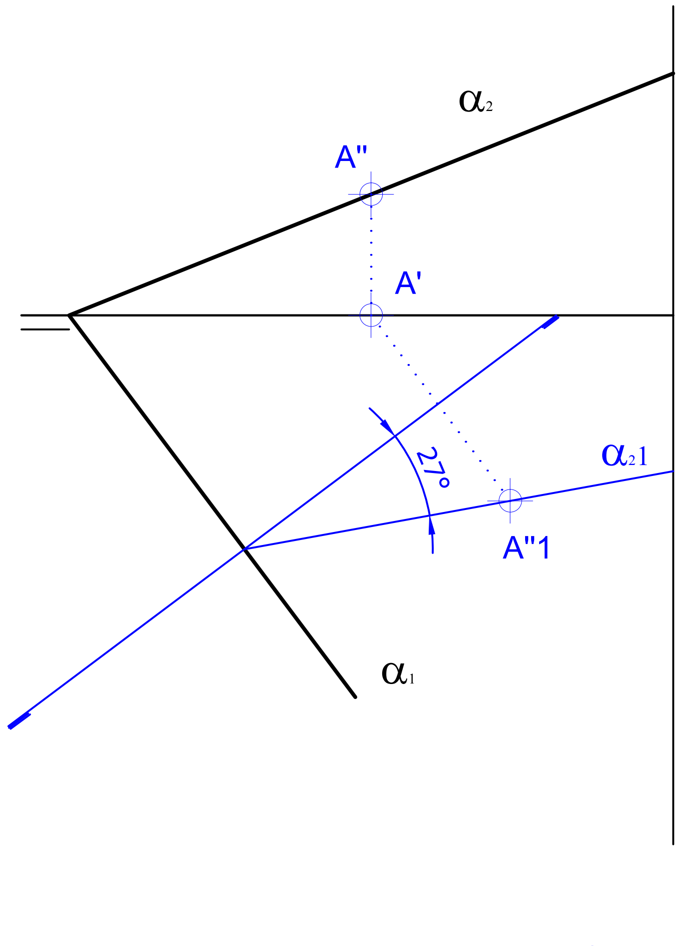
EXERCISE 1

Calculate the angle between the planes $\alpha: 4x+3y+10z=32$ *and* $\beta: z=0$.

Find the angle between the plane α and the PH.



This exercise has been solved by changing planes, by means of the vertical projection.

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EXERCISE 1

Calculate the angle between the planes $\alpha : 4x + 3y + 10z = 32$ *and* $\beta : z = 0$.

Solution:

$\vec{n}_\alpha = (4,3,10)$ and $\vec{n}_\beta = (0,0,1)$ are the normal vectors of the planes. The angle formed by both planes is given by the following expression:

$$\theta = \arccos\left(\frac{\left|\vec{n}_\alpha \cdot \vec{n}_\beta\right|}{\left|\vec{n}_\alpha\right|\left|\vec{n}_\beta\right|}\right) \Rightarrow \theta = \arccos\left(\frac{\left|4\cdot 0 + 2\cdot 0 + 10\cdot 1\right|}{\sqrt{4^2+3^2+10^2}\cdot\sqrt{0^2+0^2+1^2}}\right) = \arccos\left(\frac{2}{\sqrt{5}}\right)$$

$$\theta = 26,5650^\circ$$

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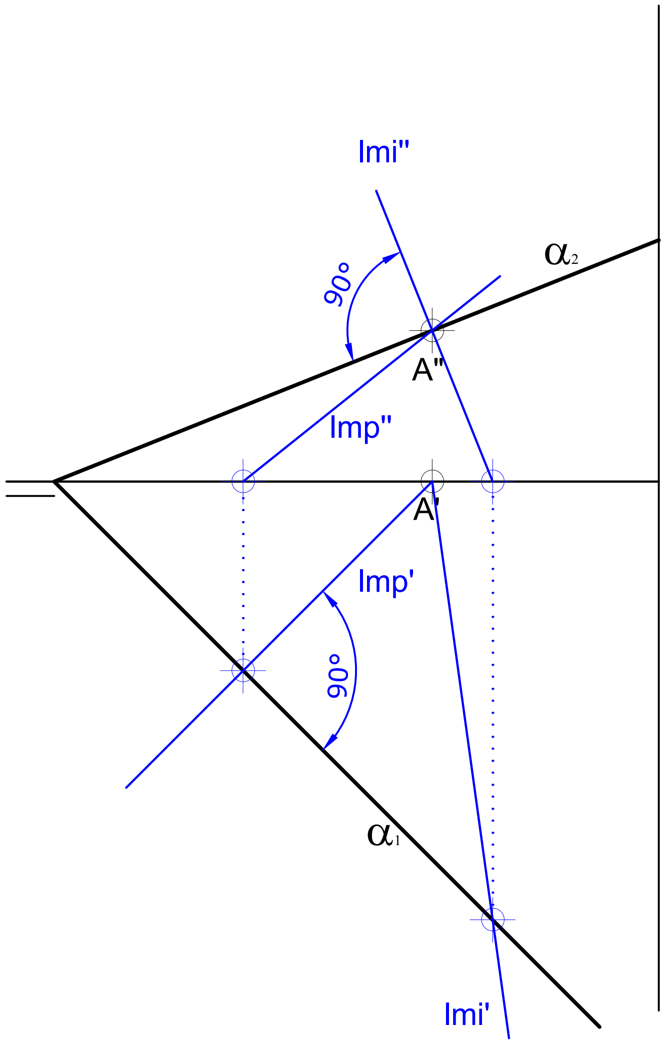
EXERCISE 2

Calculate the lines of maximum slope and maximum inclination of the plane $\alpha: 2x+2y+5z=16$ *in the point* $A(3,0,2)$.

Draw from the point A the line of maximum slope and the line of maximum inclination of the plane α .



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EXERCISE 2

Calculate the lines of maximum slope and maximum inclination of the plane $\alpha: 2x+2y+5z=16$ *in the point* $A(3,0,2)$.

<u>*Solution:*</u>

The procedure to obtain the line of maximum slope of the plane α in the point A is the following:

- Calculate the line of intersection r between the planes α and XOY:

$$r:\begin{cases} 2x+2y+5z=16 \\ z=0 \end{cases}$$

- Calculate the plane π, that passing through the point A is perpendicular to r:

As the plane π is perpendicular to the line r, the normal vector of the plane and the direction vector of the line r are parallel. We will calculate the direction vector of r:

$$\vec{v}_r = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 5 \\ 0 & 0 & 1 \end{vmatrix} = 2\vec{i} - 2\vec{j} = (2,-2,0)$$

Next, we calculate the plane π with normal vector $\vec{n}_\pi = (2,-2,0)$ and passing through the point $A=(3,0,2)$:

$$\pi: 2(x-3)-2(y-0)+0(z-2)=0 \Rightarrow \pi: x-y-3=0$$

- The line of maximum slope is the intersection between the planes α and π:

$$\begin{cases} 2x+2y+5z=16 \\ x-y=3 \end{cases}$$

The procedure to obtain the line of maximum inclination of the plane α in the point A is similar. Only the first step is different from the procedure described before:

- Calculate the line of intersection s between the planes α and XOZ:

$$s:\begin{cases} 2x+2y+5z=16 \\ y=0 \end{cases}$$

EXERCISE 2

- Calculate the plane β, that passing through the point A is perpendicular to s:

As the plane β is perpendicular to the line s, the normal vector of the plane and the direction vector of the line are parallel. We will calculate the direction vector of s:

$$\vec{v}_s = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 5 \\ 0 & 1 & 0 \end{vmatrix} = -5\vec{i} + 2\vec{k} = (-5, 0, 2)$$

Next, we calculate the plane β with normal vector $\vec{n}_\beta = (-5, 0, 2)$ and passing through the point $A = (3, 0, 2)$:

$$\beta: -5(x-3) + 0(y-0) + 2(z-2) \Rightarrow \beta: 5x - 2y - 11 = 0$$

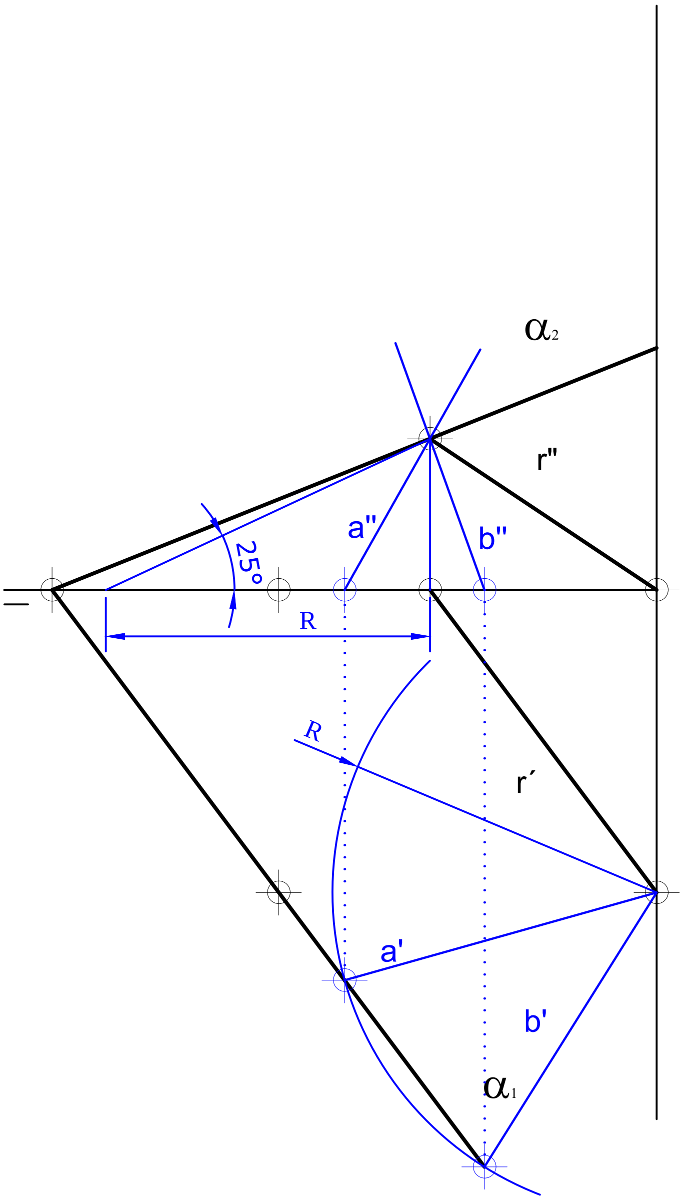
- The line of maximum inclination is the intersection between the planes α and β:

$$\begin{cases} 2x + 2y + 5z = 16 \\ 5x - 2y = 11 \end{cases}$$

EXERCISE 3

Calculate the lines that intersect the line $r: \frac{x}{3} = \frac{y-4}{-4} = \frac{z}{2}$, are included in the plane defined by the points $(8,0,0)$, $(3,0,2)$ and $(5,4,0)$, and form an angle of 25º with the plane XOY.

Define and draw the lines that intersect the line r, are included in the plane α, and their angle with the PH is 25º.



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EXERCISE 3

Calculate the lines that intersect the line $r: \frac{x}{3} = \frac{y-4}{-4} = \frac{z}{2}$, are included in the plane defined by the points $(8,0,0)$, $(3,0,2)$ and $(5,4,0)$, and form an angle of 25º with the plane XOY.

Solution:

We will start by calculating the plane π determined by the points $(8,0,0)$, $(3,0,2)$ and $(5,4,0)$. We need two non-parallel vectors. For example, we choose these vectors:

$$\overrightarrow{AB} = (3,0,2) - (8,0,0) = (-5,0,2)$$
$$\overrightarrow{BC} = (5,4,0) - (3,0,2) = (2,4,-2)$$

And the plane π is given by:

$$\begin{vmatrix} x-8 & -5 & 2 \\ y & 0 & 4 \\ z & 2 & -2 \end{vmatrix} = 0 \Rightarrow \pi : 4x + 3y + 10z - 32 = 0$$

Next we calculate the line that intersecting the line r is included in the plane π and forms an angle of 25º with the plane XOY. The plane XOY is given by $z=0$, being its normal vector $\vec{n} = (0,0,1)$. Let $\vec{v}_s = (a,b,c)$ be the direction vector of the requested line.
The normal vector of the plane π is $\vec{n}_\pi = (4,3,10)$. As the searched line is included in the plane π, the following is satisfied:

$$\vec{v}_s \perp \vec{n}_\pi \Rightarrow \vec{v}_s \cdot \vec{n}_\pi = 0 \Rightarrow 4a + 3b + 10c = 0 \Rightarrow c = -\frac{4a+3b}{10}$$

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On the other hand, the searched line forms an angle of 25º with the XOY plane:

$$\sin\alpha = \frac{|\vec{v}_s \cdot \vec{n}|}{|\vec{v}_s||\vec{n}|} \Rightarrow \sin 25^\circ = \frac{\left|-\frac{4a+3b}{10}\right|}{\sqrt{a^2+b^2+\left(\frac{4a+3b}{10}\right)^2}} \Rightarrow 0,42 = \frac{\left|-\frac{4a+3b}{10}\right|}{\sqrt{a^2+b^2+\left(\frac{4a+3b}{10}\right)^2}} \Rightarrow$$

$$0,42^2 = \frac{\left(\frac{4a+3b}{10}\right)^2}{a^2+b^2+\left(\frac{4a+3b}{10}\right)^2} \Rightarrow 0,1764\left(a^2+b^2+\left(\frac{4a+3b}{10}\right)^2\right) = \left(\frac{4a+3b}{10}\right)^2 \Rightarrow$$

$$0,1764\left(100a^2+100b^2+(4a+3b)^2\right) = (4a+3b)^2 \Rightarrow$$

$$3,8224a^2 - 19,7666ab + 9,587b^2 = 0 \Rightarrow$$

$$a = \frac{19,7666b \pm \sqrt{(19,7666b)^2 - 4 \cdot 3,8224 \cdot 9,5876b^2}}{2 \cdot 3,8224} \Rightarrow a = \begin{cases} 0,5418b \\ 4,6292b \end{cases}$$

Hence, there are two lines that satisfy the requested conditions, being their direction vectors: $\vec{v}_{s1} = \left(0,5418b, b, -\frac{4 \cdot 0,5418b + 3b}{10}\right)$ and $\vec{v}_{s1} = \left(4,6292b, b, -\frac{4 \cdot 4,6292b + 3b}{10}\right)$.

As the searched line, intersects the line r, the line will be completely determined by obtaining this point of intersection. The point of intersection between the searched line and the line r, is the point of intersection of r with the plane π. Let $P(3\lambda, 4-4\lambda, 2\lambda)$ be the expression of a generic point of the line r. The intersection between r and the plane π is:

$$4 \cdot 3\lambda + 3 \cdot (4-4\lambda) + 10 \cdot 2\lambda - 32 = 0 \Rightarrow 20\lambda - 20 = 0 \Rightarrow \lambda = 1$$

Therefore, $Q(3,0,2)$ is the intersection point between r and π. Which means that $Q(3,0,2)$ is the intersection point between the searched line and r.

And these two lines are obtained:

$$s1: \begin{cases} x = 3 + 0,5418\lambda \\ y = \lambda \\ z = 2 - 0,51672\lambda \end{cases} \quad \text{and} \quad s2: \begin{cases} x = 3 + 4,6292\lambda \\ y = \lambda \\ z = 2 - 2,1516\lambda \end{cases}$$