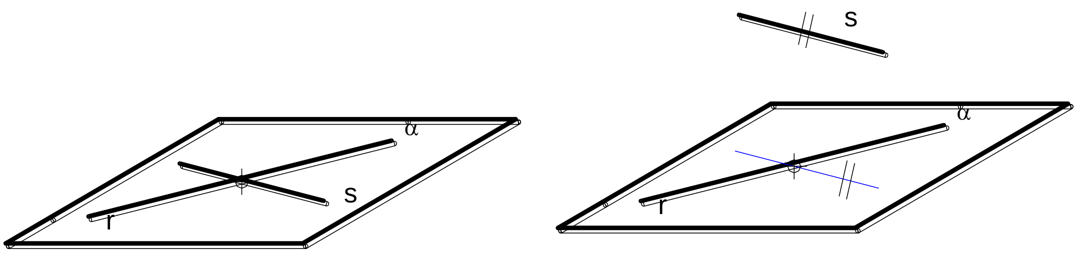
LESSON VI: ANGLES BETWEEN ELEMENTS

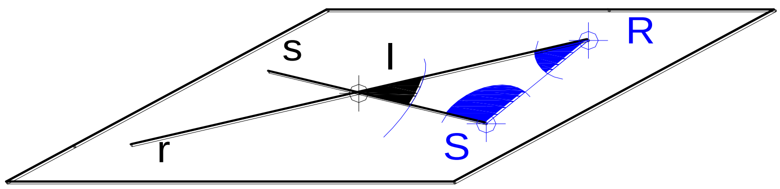
6.1.G – Angle between two lines

It is easy to calculate the angle between two lines that intersect. The angle between two skew lines is the angle that one of the lines form with the line that passing through a point of the first line is parallel to the second line.



In orthogonal projections systems, there are different processes to calculate the angle between two lines.

From the strict point of view of the geometry, the angle between two lines can be obtained by calculating the angles of a RIS triangle and the length of its sides: **R** is any point of the line **r**, **S** is any point of the line **s** and **I** is the point of intersection of both lines.



From the point of view of the descriptive geometry, the problem of calculating the angle between two lines is based on having a plane that contains both lines with an adequate position in order to see clearly the angle formed by the lines. These three processes can be followed to obtain this angle:

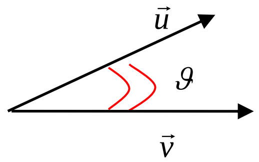
1) Foldings

2) Changing planes

3) Rotations

6.1.A – Angle between two lines

Remember that the angle between two vectors, $\vec{u}$ and $\vec{v}$,



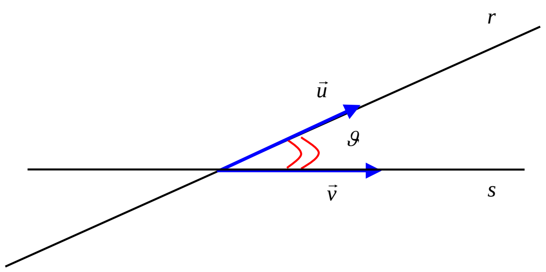
can be calculated by the expression $\cos\vartheta = \frac{|\vec{u}\cdot\vec{v}|}{|\vec{u}|\cdot|\vec{v}|}$.

Given two lines the possibilities for their relative positions in the space are: the lines coincide with each other, they are parallel, they have exactly one point of intersection or they are skew. Depending on these possibilities, the angle between two lines can be:

- Two lines with exactly one point of intersection: In this case, both lines determine four angles, being the vertically opposite angles equal. The smallest angle from these two different angles is the angle between the two lines.
- Skew lines: Taking one of the lines, a line that is parallel to this first line and intersects the other line is determined. The smallest angle formed in the intersection of these lines is the angle between the given lines.
- When the given lines coincide or they are parallel, the angle between them is 0º.

The angle between lines is:

- the angle between direction vectors of the lines, when the angle is acute.
- the supplementary angle between direction vectors of the lines, when the angle is obtuse.



Hence, the angle between the lines r and s, with direction vectors $\vec{u}=(u_1,u_2,u_3)$ and $\vec{v}=(v_1,v_2,v_3)$ respectively, is given by the following formula:

$$\theta = ang(r,s) = \arccos\frac{|\vec{u}\cdot\vec{v}|}{|\vec{u}|\cdot|\vec{v}|} = \arccos\frac{|u_1v_1+u_2v_2+u_3v_3|}{\sqrt{u_1^2++u_2^2+u_3^2}\cdot\sqrt{v_1^2++v_2^2+v_3^2}}$$

If the angle obtained in this way is obtuse, its supplementary is taken:

$$\cos(180-\vartheta) = -\cos\vartheta$$

As a particular case, r and s are orthogonal if $\cos\theta = 0 \Rightarrow$ that is to say, if $u_1v_1+u_2v_2+u_3v_3=0$.

6.1. Mutual examples of both subjects

►*Example 38* (A)

Find the angle between the lines $r:\begin{cases} x-3y=1 \\ 2y=z \end{cases}$ and $s:\begin{cases} x+3z=10 \\ y-2z=-3 \end{cases}$.

Solution: By considering two points in each of the lines (r and s) the direction vectors of the lines will be calculated. We will consider the points A=(1,0,0) and B=(4,1,2) in the line r, its direction vector is $\vec{u}=(3,1,2)$.

If we choose the points C=(4,1,2) and D=(1,3,3) in the line s, we obtain its direction vector $\vec{v}=(3,-2,-1)$. Therefore,

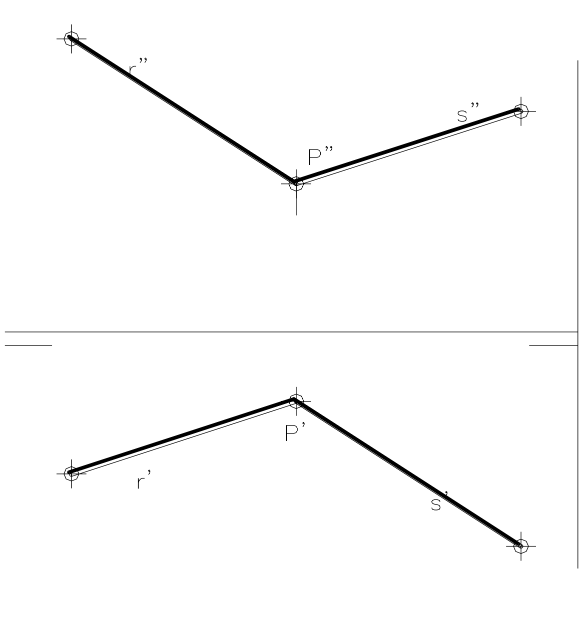
$$\theta = arcos\frac{|\vec{u}.\vec{v}|}{|\vec{u}|.|\vec{v}|} = arcos\frac{|(3,1,2).(3,-2,-1)|}{\sqrt{9+1+4}.\sqrt{9+4+1}} = arcos\frac{|9-2-2|}{\sqrt{14}.\sqrt{14}} = arcos\frac{5}{14}$$

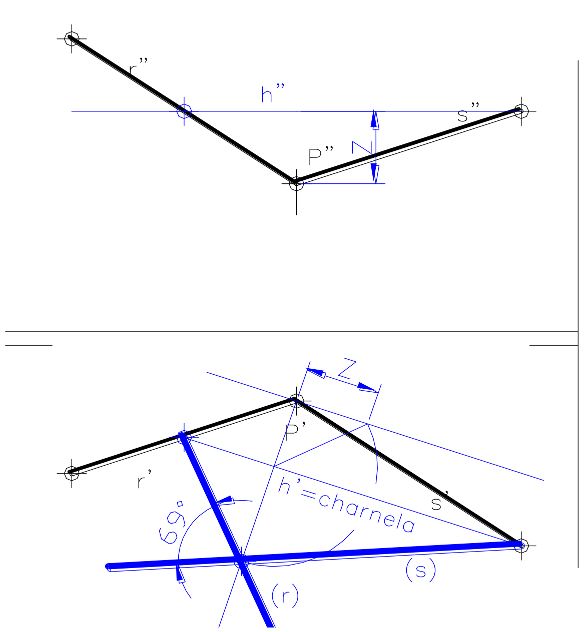
$$= arcos\ 0{,}3771 \rightarrow \boxed{\theta = 69{,}08°}$$

►*Example 38* (G)

Find the angle between the lines r and s.

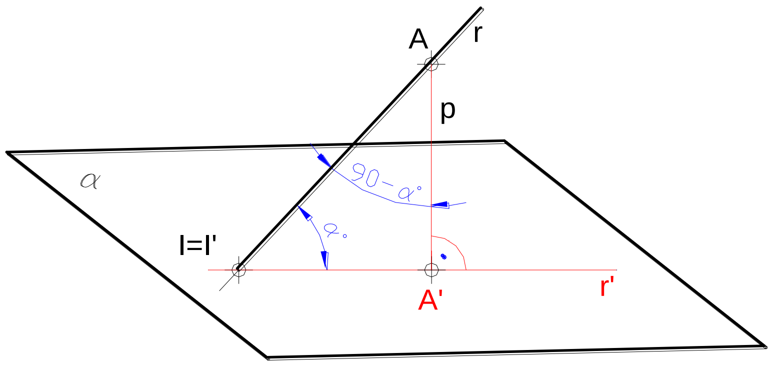
Solution: As the given lines intersect each other, they form a plane. We have to fold this plane in order to find the angle between both lines. In this case, the horizontal h has been used as the axis in the fold.





6.2.G – Angle between a line and a plane

The angle between a line and a plane is the angle formed by the line and its orthogonal projection in the plane. As it can be observed in the figure, this angle is the complementary of the angle formed by the line r and the line that passing through the point is perpendicular to the plane. Therefore, the problem is reduced to the previous case.



6.2.A – Angle between a line and a plane

Given a line and a plane, the possibilities for their relative positions in the space are: the line is included in the plane, they are parallel, they have exactly one point of intersection. Depending on these possibilities, the angle between them can be:

- The line and the plane intersect in one point: the angle between them is the one formed by the line and its orthogonal projection in the plane (which is the line of intersection between the given plane and the plane that containing the given line is perpendicular to the given plane).

- The line is included in the plane or they are parallel: the angle between them is 0º.

The angle ϑ between the line r and the plane π, is the complementary of the angle formed by r with a perpendicular line s to the plane π.

The angle between the lines r and s is:

- the angle between the direction vector of r and the normal vector of the plane π, when the angle is acute.
- the supplementary angle between the direction vector of r and the normal vector of the plane π, when the angle is obtuse.

Hence, the angle between the line r and the plane $\pi: Ax+By+Cz+D=0$, with direction vector $\vec{u}=(u_1,u_2,u_3)$ and normal vector $\vec{a}$ respectively, is given by the following formula:

$$\theta = ang(r,\pi) = \arcsin\frac{|\vec{u}\cdot\vec{a}|}{|\vec{u}|\cdot|\vec{a}|} = \arcsin\frac{|Au_1+Bu_2+Cu_3|}{\sqrt{u_1^2++u_2^2+u_3^2}\cdot\sqrt{A^2+B^2+C^2}}$$

Take into account that the following equality is satisfied: $\cos(90-\vartheta)=sen\vartheta$.

As a particular case, the line r and the plane π are perpendicular if $sen\theta=0 \Rightarrow \vec{u}$ and $\vec{a}$ are parallel $\Rightarrow \frac{A}{u_1}=\frac{B}{u_2}=\frac{C}{u_3}$

►*Example 39* *(A)*

Find the angles that the line $r:\begin{cases} 2x+5z=24 \\ y=z \end{cases}$ form with the horizontal plane $(z=0)$ and the vertical plane $(y=0)$.

Solution: Choosing two points in the line r, A=(12,0,0) and B=(2,4,4), we can calculate the direction vector of r: $\vec{u}=(5,-2,-2)$. The normal vectors of the planes $y=0$ and $z=0$ are respectively $\vec{a}=(0,0,1)$ and $\vec{b}=(0,1,0)$.

- The angle between the line r and the horizontal plane is:

$$\theta = arcsin\ \frac{|\vec{u}.\vec{a}|}{|\vec{u}|.|\vec{a}|} = arcsin\frac{|(5,-2,-2).(0,0,1)|}{\sqrt{25+4+4}\,.\sqrt{1}} = arsin\frac{|-2|}{\sqrt{33}} = arcsin\ 0{,}3481$$

$$\boxed{\theta = 20{,}37°}$$

- The angle between the line r and the vertical plane is:

$$\theta = arcsin\ \frac{|\vec{u}.\vec{a}|}{|\vec{u}|.|\vec{a}|} = arcsin\frac{|(5,-2,-2).(0,0,1)|}{\sqrt{25+4+4}\,.\sqrt{1}} = arcsin\ 0{,}3481 \rightarrow \boxed{\theta = 20{,}37°}$$

6.2. Mutual examples of both subjects

►*Example 40* *(A)*

Find the angle between the line defined by the points (7,2,2) and (2,4,4), and the plane defined by the points (5,1,4), (1,3,2) and (7,4,1).

Solution: The direction vector of the line that passes through the points (7,2,2) and (2,4,4*)* is $\vec{u}=(5,-2,-2)$.

And the equation of the plane defined by the three points (5,1,4), (1,3,2) and (7,4,1) is $y+z=5$, having as normal vector $\vec{b}=(0,1,1)$.

Therefore, the angle between the line and the plane is:

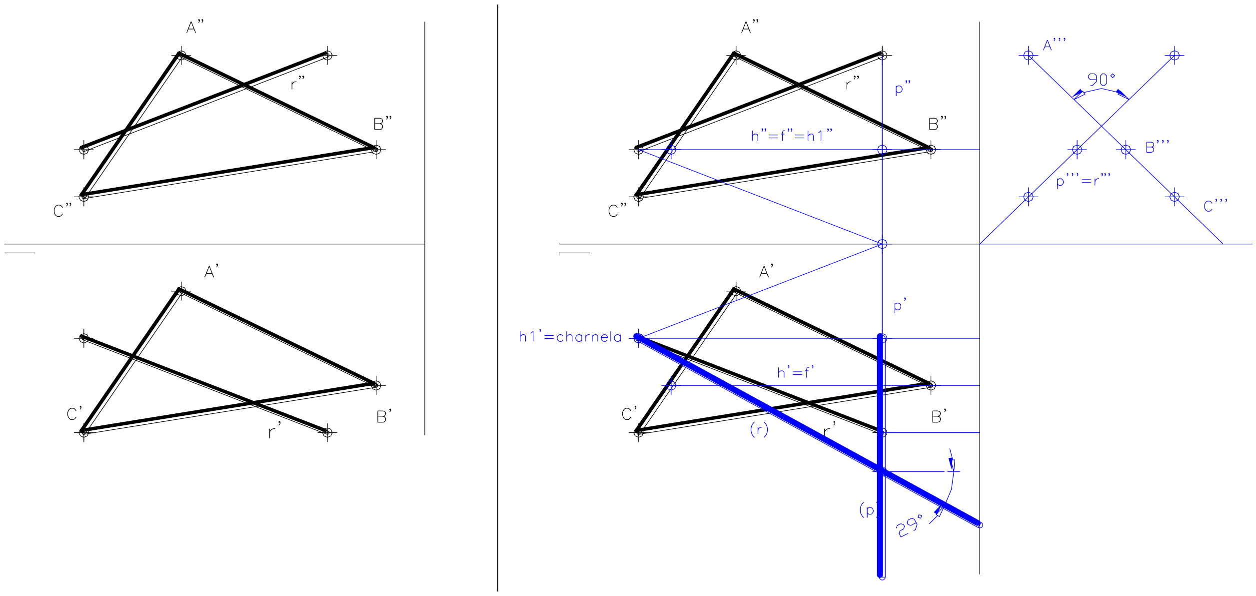
$$\theta = arcsin\ \frac{|\vec{u}.\vec{b}|}{|\vec{u}|.|\vec{b}|} = arcsin\frac{|(5,-2,-2).(0,1,1)|}{\sqrt{25+4+4}\,.\sqrt{1+1}} = arcsin\frac{|-2-2|}{\sqrt{33}\,.\sqrt{2}} = arcsin\frac{|-4|}{\sqrt{66}}$$

$$= arcsin\ 0{,}4924 \rightarrow \boxed{\theta = 29{,}49°}$$

►*Example 40* (G)

Find the angle between the line r and the plane defined by the points ABC.

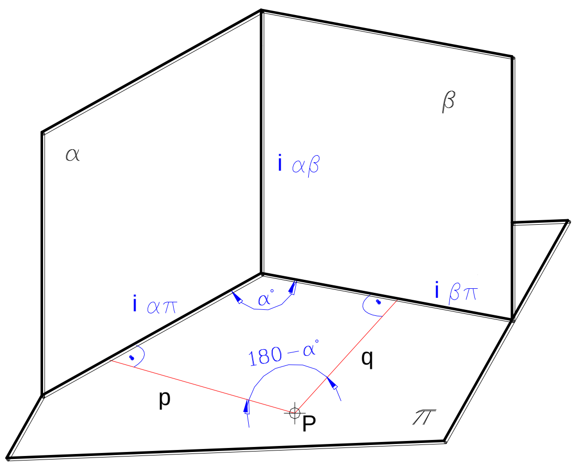
Solution: A point of the line r is chosen and a perpendicular line p to the plane passing through the chosen point is drawn. Both lines define a plane. As in a previous exercise, we have to fold this plane and the angle between the folded lines is calculated.



6.3.G – Angle between two planes

The angle between two planes is the angle between the intersection lines of these planes with a third plane which is perpendicular to the given planes. This plane can be defined by drawing from any point two perpendicular lines to the planes.

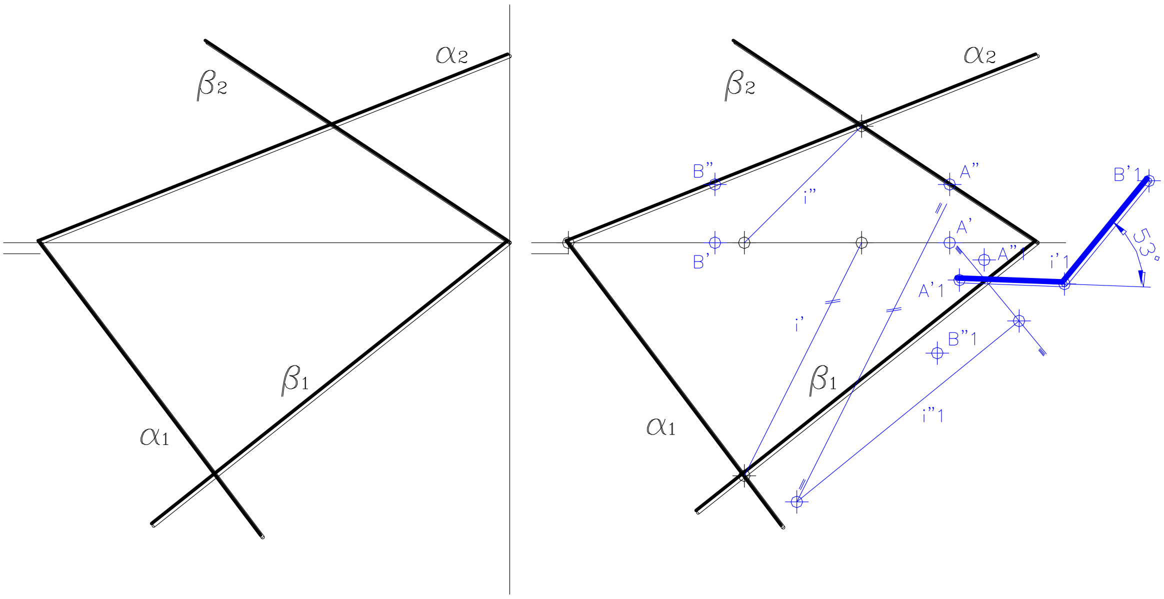
As it can be observed in the next figure, the angle between these lines is the supplementary angle of the lines of intersection. Taking into account these considerations, the angle between two planes is reduced to the case of calculating the angle between two lines (6.1.G).



►Example 41 (G)

Find the angle between the planes α and β.

Solution: This exercise has been solved by setting the plane defined by p and q parallel to a projection plane. This is the same as finding the line of intersection (i) between both planes and making it perpendicular to a projection plane (PH1) by changing planes. In this projection, the angle between both lines can be measured directly.



6.3.A – Angle between two planes

Given two planes, the possibilities for their relative positions in the space are: the planes coincide, they are parallel or they intersect each other in a line.

Depending on these possibilities, the angle between them can be:

- Two planes that intersect: In this case, both planes determine four angles, being the vertically opposite angles equal. The smallest angle from these two different angles is the angle between the two planes.

- When the given planes coincide or they are parallel, the angle between them is 0º.

If the angle formed by the plane is acute, the angle between two planes is the angle formed by their normal vectors. If the angle formed by the normal vectors is obtuse, the angle between two planes is the supplementary angle of the normal vectors.

Therefore, given the planes $\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$ and $\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$ with normal vectors $\vec{a}_1$ and $\vec{a}_2$ respectively, the angle between both planes is given by the following expression:

$$\theta = ang(\pi_1, \pi_2) = \arccos \frac{|\vec{a}_1 \cdot \vec{a}_2|}{|\vec{a}_1| \cdot |\vec{a}_2|} = \arccos \frac{|A_1A_2 + B_1B_2 + C_1C_2|}{\sqrt{A_1^2 + +B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + +B_2^2 + C_2^2}}$$

If the angle obtained in this way is obtuse, its supplementary angle will be used:

$\cos(180-\vartheta)=-\cos\vartheta$

As a particular case, π_1 and π_2 are perpendicular if $\cos\theta=0 \quad \Rightarrow$ if $A_1A_2+B_1B_2+C_1C_2$

Given the planes $\pi_1: A_1x+B_1y+C_1z+D_1=0$ and $\pi_2: A_2x+B_2y+C_2z+D_2=0$, the set of equidistant points from both planes π_1 and π_2 $(d(P,\pi_1)=d(P,\pi_2))$ is given by:

$$\frac{A_1x+B_1y+C_1z+D_1}{\sqrt{A_1^2++B_1^2+C_1^2}}=\frac{A_2x+B_2y+C_2z+D_2}{\sqrt{A_2^2++B_2^2+C_2^2}}$$

$$\frac{A_1x+B_1y+C_1z+D_1}{\sqrt{A_1^2++B_1^2+C_1^2}}=-\frac{A_2x+B_2y+C_2z+D_2}{\sqrt{A_2^2++B_2^2+C_2^2}}$$

These planes are called bisector planes of π_1 and π_2.

►***Example 42*** **(A)**

Find the angle between the plane $\pi: 4x-5y-6z=0$ and the vertical plane $y=0$.

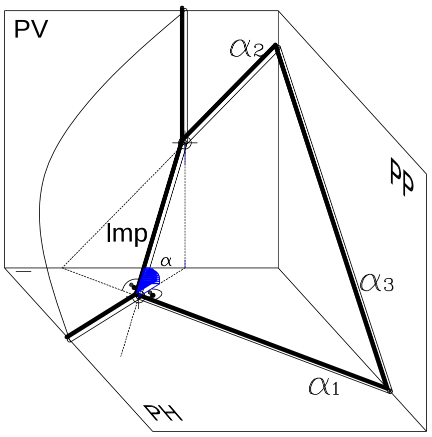
Solution: The normal vectors of the planes π and $y=0$, are respectively $\vec{a}=(4,-5,-6)$ and $\vec{b}=(0,1,0)$. And the angle between the given planes is:

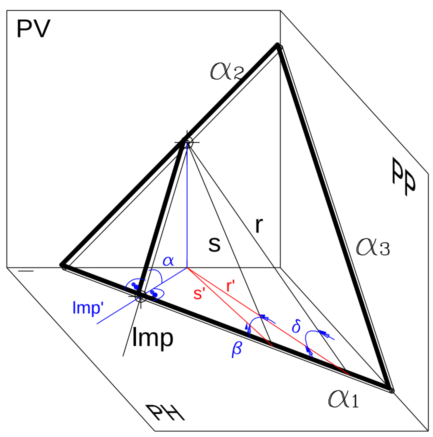
$$\theta = arcos\ \frac{|\vec{a}.\vec{b}|}{|\vec{a}|.|b|}=arcos\frac{|(4,-5,-6).(0,1,0)|}{\sqrt{16+25+36}.\sqrt{1}}=arcos\frac{|-5|}{\sqrt{77}}=arcos\ 0{,}5698$$

$$\rightarrow \boxed{\theta = 55{,}26°}$$

6.4.G – Angles with the projection planes

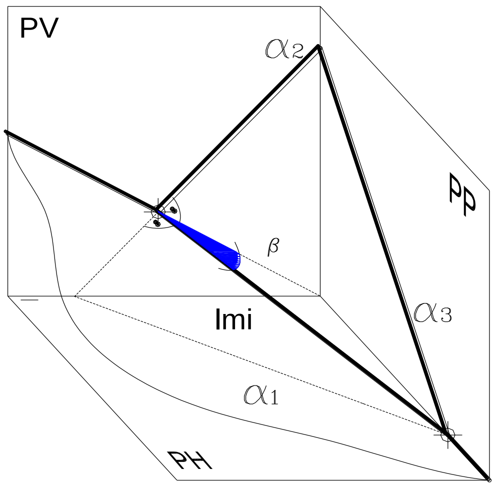
The angle between a plane and the plane PH can be calculated by drawing a plane that is perpendicular to the intersection of both planes (**α**1). The auxiliary plane intersects the plane **α** in a line that is called the line of maximum slope (**lmp**). This is the line of the plane that forms the largest angle with the plane PH, and this line is enough to define the plane. It is perpendicular to the plane **α**1. See the attached figures.

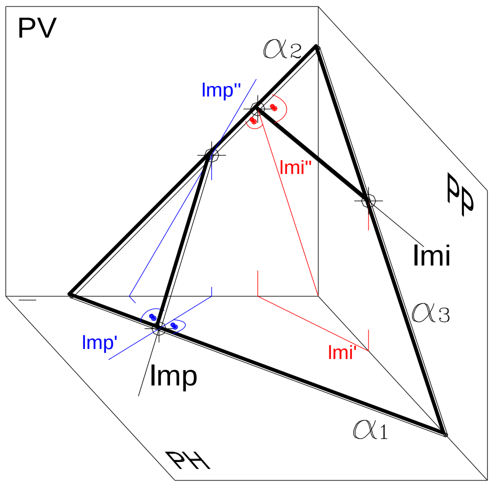




Given a point of the plane, in order to draw the line of maximum slope that passes through this point, it has to be taken into account that its projection in the PH plane is perpendicular to the plane $\boldsymbol{\alpha}_1$.

In the same way, the angle between the plane with the PV and the line of maximum inclination (**Imi**) are defined. See figures. In this case, the projection of this line in the PV plane is perpendicular to the plane $\boldsymbol{\alpha}_2$.

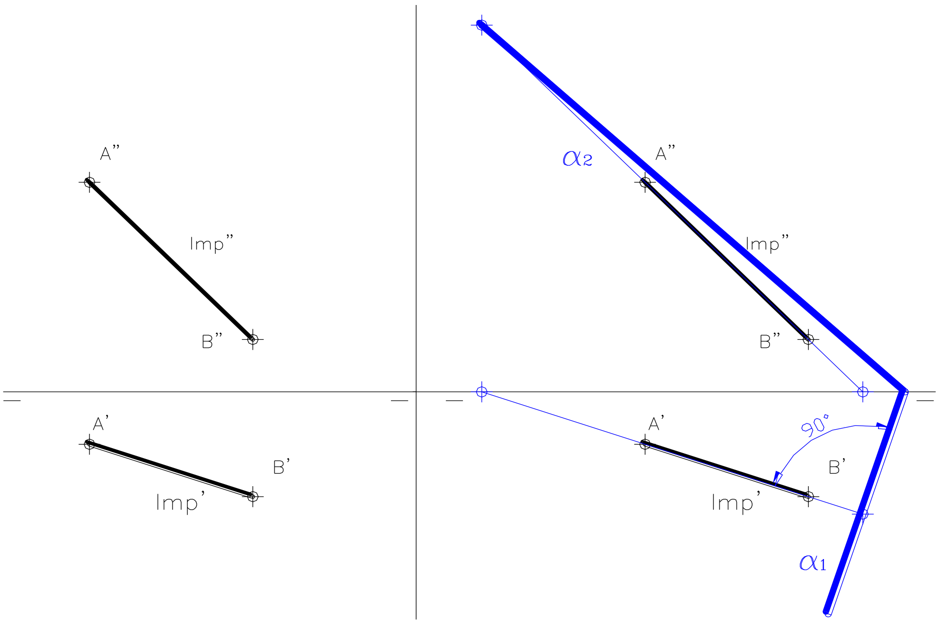




►*Example 43* *(G)*

Determine the plane α, knowing its **Imp**.

Solution: As the **Imp** is the line of maximum slope, the horizontal trace of the plane is perpendicular to it. In the following figure, after having calculated both traces of the line, the plane α_1 has been drawn from the horizontal trace.



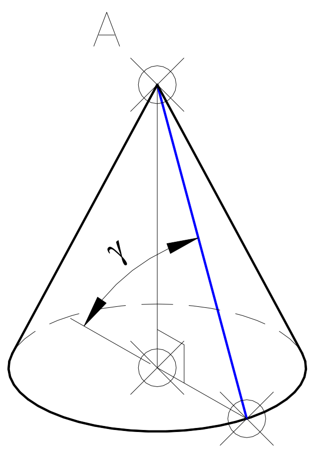
6.5.G – Inverse problems

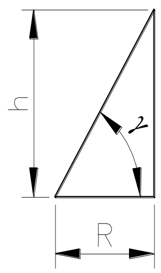
Sometimes it is known the angle between two elements. In these cases, making use of the angle and other conditions the elements have to be determined. These type of problems are called “inverse problems”.

Basic concepts

1) The geometric place of the lines that passing through a given point form a certain angle with another plane is a cone with the following features:

- The point that is known is the vertex c (point A in the figure).
- The cone obtained in this way is a stra being its axis perpendicular to the plane.
- The basis of the cone is circular, and its depends on its height (h) and the known angle



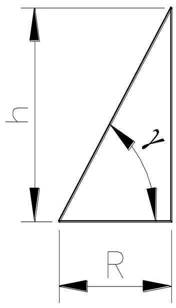


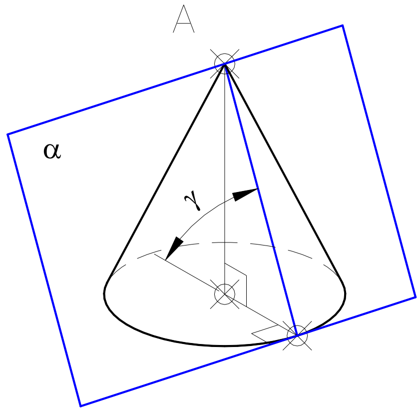
All the generatrix line of this cone are solutions of the problem.

2) The planes that passing through a given point form a certain angle with another plane, are tangent to a cone with the following features:

- The vertex of the cone is the known point (point A in the figure).
- The cone is a straight cone, being its axis perpendicular to the plane.

- The basis of the cone is circular, and its radius (R) depends on its height (h) and the known angle (γ).



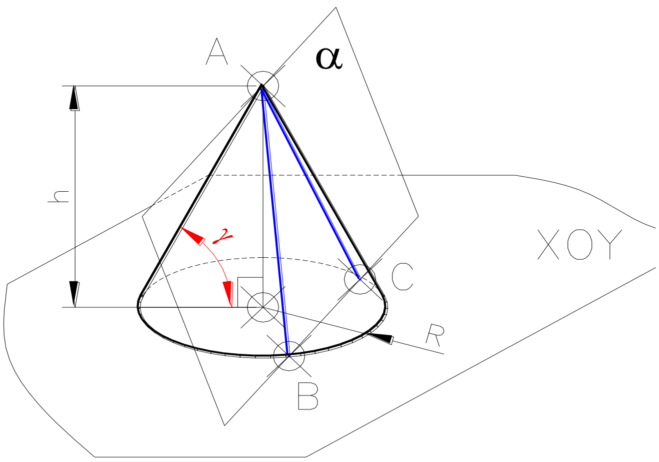


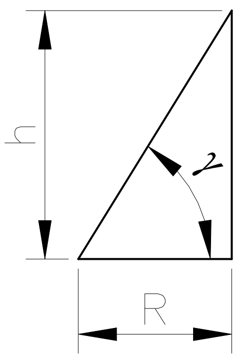
Problem of the type 1:

Determine all the lines that passing through a point A, are included in the plane α and form an angle γ with the XOY plane.

The resolution method consists in:

1. Calculating the dimensions of the cone (the radius R of the basis): its vertex will be the point A, the height of the cone (h) will be the Z coordinate.
2. Drawing the basis of the cone: circumference of radius R and center A'.
3. Calculating the intersection of the plane α with the basis of the cone: points B and C.
4. The two solutions of the general case are the lines AB and AC.



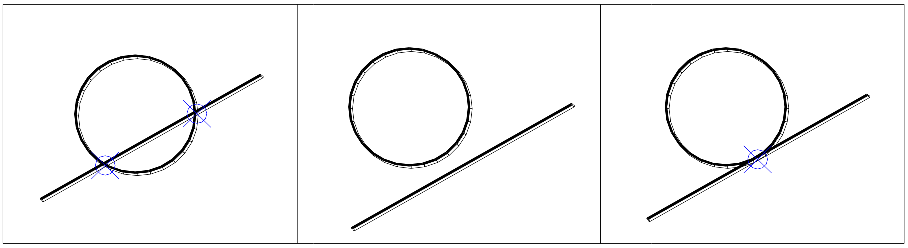


Possibilities:

a) If the slope of the requested lines is smaller than the slope of the plane, there are two solution lines (the plane will intersect the cone by two generatrix lines).

b) If the slope of the requested lines is greater than the slope of the plane, there is no solution (the plane will not intersect the plane).

c) If the slope of the requested lines is the same as the slope of the plane, there is one solution line (the plane will be tangent to the cone).

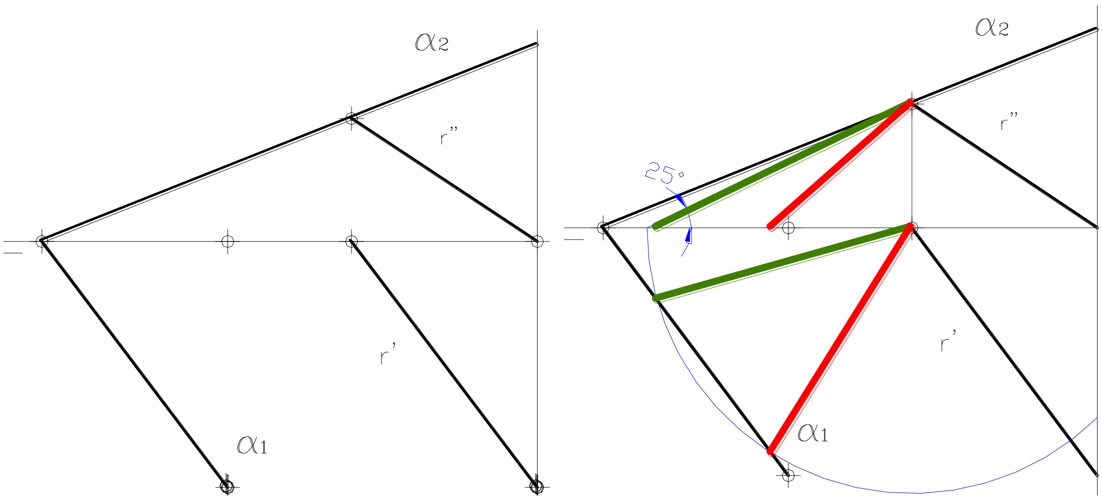


a) b) c)

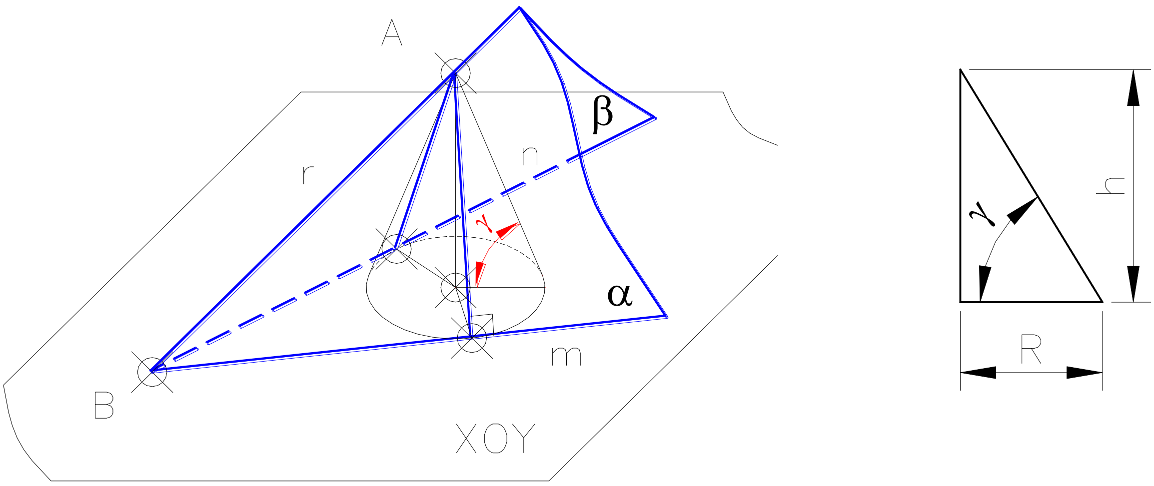
►*Example 44* (G)

Determine and draw the lines that intersect the line r, are included in the plane α and form an angle of 25º with the PH (XOY) plane.

Solution: Two lines conform the solution.



Problem of the type 2: Determine all the planes that contain the line r forming an angle γ with the plane XOY.



The resolution method consists in:

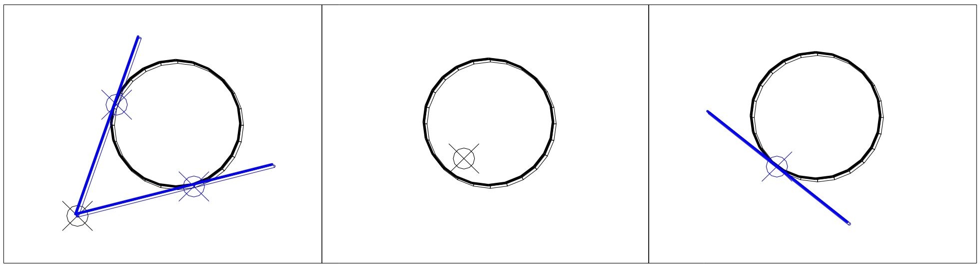
1. Calculating the dimensions of the cone (the radius R of the basis): its vertex will be any point of the line r (point A in the figure), the height of the cone (h) will be the Z coordinate.
2. Drawing the basis of the cone: circumference of radius R and center A'.
3. Calculating the horizontal trace of the line r: point B.
4. Drawing from B the tangent lines to the basis of the cone: horizontal lines m and n.
5. The two solutions of the general case are the planes α = r+m, and β = r+n.

Possibilities:

a) If the slope of the requested planes is greater than the slope of the line, there are two solution planes (two planes will be tangent to the cone).

b) If the slope of the requested planes is smaller than the slope of the line, there is no solution (there is no tangent plane to the cone).

c) If the slope of the requested planes is the same as the slope of the line, there is one solution (one plane will be tangent to the cone).



a) b) c)

6.5. Mutual examples of both subjects

►***Example 45 (A)***

Find the plane that contains the line $r\colon\begin{cases} x - z = 3 \\ 3y + z = 9 \end{cases}$ and forms an angle of 45º with the plane XOY.

Solution:

Using the concept of sheaf of planes, it is known that the plane that contains the line r is:

$\pi_1\colon x - z - 3 + \alpha\,(3y + z - 9) = 0$, being its normal vector $n_{\pi_1} = (1, 3\alpha, \alpha - 1)$.

On the other hand, the normal vector of the plane $\pi_2\colon z = 0$ is $n_{\pi_2} = (0,0,1)$. As the angle between two planes is the angle between their normal vectors:

$$\cos 45° = \frac{|(1,3\alpha,\alpha-1)(0,0,1)|}{\sqrt{1^2 + (3\alpha)^2 + (\alpha-1)^2}\ \sqrt{1}} = \frac{|\alpha - 1|}{\sqrt{10\alpha^2 - 2\alpha + 2}} = \frac{1}{\sqrt{2}} \quad \rightarrow$$

$$2(\alpha-1)^2 = 10\alpha^2 - 2\alpha + 2 \rightarrow \; 4\alpha^2 + \alpha = 0 \; \rightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = -\frac{1}{4} \end{cases}$$

By substituting these values in the plane π_1, the following two planes are obtained:

$\pi_{1a}: x - z - 3$ and $\pi_{1b}: 4x - 3y - 5z - 3 = 0$

►*Example 45* (G)

Find and draw the planes that contain the line r and form an angle of 45º with the plane PH.

Solution: There are two planes, α and β, that satisfy the requested conditions.