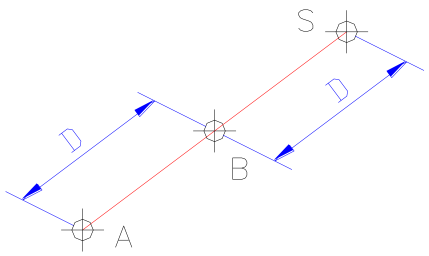
LESSON V: SYMMETRIES

Given a point three different types of symmetries will be considered:

1. Symmetry with respect to a point
2. Symmetry with respect to a line
3. Symmetry with respect to a plane

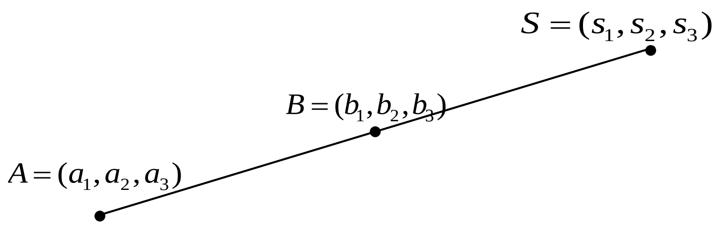
5.1.G – Symmetric point with respect to another point

Given a point (A), its symmetric with respect to another point (B) is a point (S) which is located in the line that passes through the points A and B, a distance AB from the point B.



5.1.A – Symmetric point with respect to another point. Midpoint of a segment

Let $A=(a_1,a_2,a_3)$ and $S=(s_1,s_2,s_3)$ be two points. These two points A and S are symmetric with respect to another point B, if they are the end points of the segment AS with midpoint B.



The coordinates of the midpoint B are $\left(\frac{a_1+s_1}{2},\frac{a_2+s_2}{2},\frac{a_3+s_3}{2}\right)$.

5.1. Mutual examples of both subjects

►*Example 36 (A)*

Calculate the symmetric point of $A=(4,3,3)$ with respect to the point $B=(0,1,6)$.

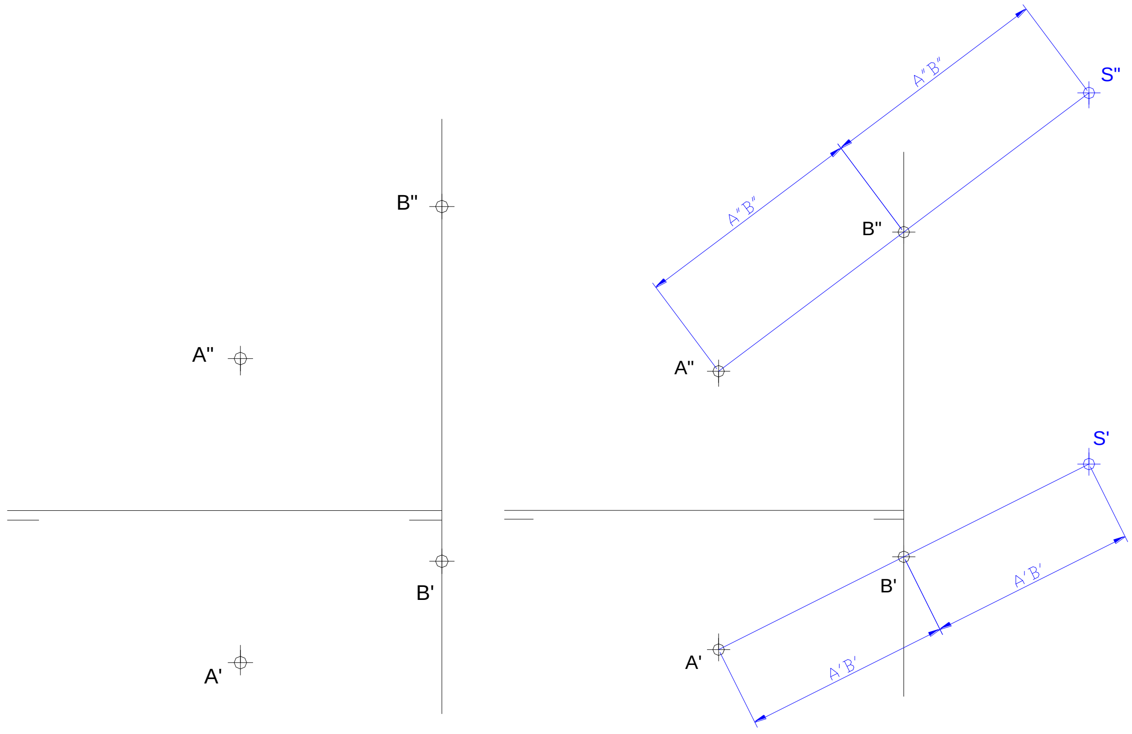
Solution: The symmetric point $S=(x,y,z)$ verifies:

$$(0,1,6)=\left(\frac{4+x}{2},\frac{3+y}{2},\frac{3+z}{2}\right) \Rightarrow S=(-4,-1,9)$$

►*Example 36 (G)*

Obtain the symmetric point of A with respect to the point B.

Solution: We join the points A and B, and the segment AB is extended a distance AB starting from the point B. In this way the symmetric point S of the point A is calculated.



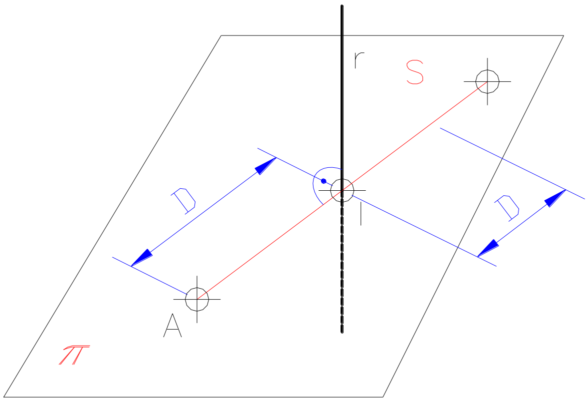
5.2.G – Symmetric point with respect to a line

Given a point (A), its symmetric point (S) with respect to a line (r) is located in a line that intersects and is perpendicular to the given line r. Assuming that the point of intersection between the lines is I, the symmetric point S is located a distance AI from the point I. It means that the symmetric point S is in a perpendicular plane to the line r, passing through A. The steps to obtain the symmetric point are the following:

1. Calculate the plane π, which is perpendicular to the line (r) and passes through the point (A)

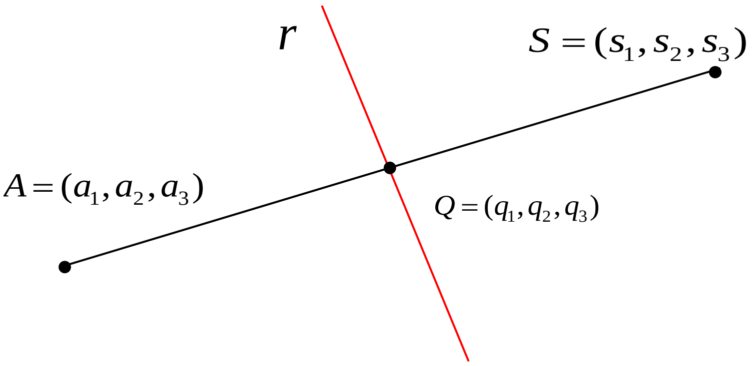
2. Calculate the point of intersection (I) between the line (r) and the plane (π)

3. Calculate the symmetric point (S) of the point A with respect to the point I



5.2.A – Symmetric point with respect to a line

Let $A=(a_1,a_2,a_3)$ and $S=(s_1,s_2,s_3)$ be two points. These points are symmetric with respect to the line r, if they are the end points of the segment AS with bisector r.



The point Q, intersection point between the segment AS and the line r, is the projection of the point A in the line r. The points A and S are symmetric with respect to the point Q. The steps to obtain the symmetric point of A with respect to the line r are the following:

1. Calculate the plane π, which is perpendicular to the line r and passes through the point A.

2. Calculate the point of intersection Q between the line r and the plane π.

3. Calculate the symmetric point S of the point A with respect to the point Q.

►*Example 37 (A)*

Calculate the symmetric point of $A=(4,2,2)$ with respect to the line $r:\frac{x-6}{4}=\frac{y-1}{2}=\frac{z-5}{-3}$.

Solution: We will calculate the plane which is perpendicular to the line r and passes through the point A. The direction vector of the line will be the normal vector of this plane: $\alpha: 4x+2y-3z+D=0$

We will make the plane pass through the point A: $4\cdot 4+2\cdot 2-3\cdot 2+D=0 \Rightarrow D=-14$

Hence, the plane is given by: $\alpha: 4x+2y-3z-14=0$.

The implicit equations of the line r are the following: $r:\begin{cases} x-2y=4 \\ 3y+2z=13 \end{cases}$

Next, we will obtain the point of intersection Q between the line r and the plane α:

$$Q=\begin{cases} x-2y=4 \\ 3y+2z=13 \\ 4x+2y-3z=14 \end{cases} \Rightarrow Q=\left(\frac{186}{29},\frac{25}{29},\frac{136}{29}\right).$$

Finally, the symmetric point of A with respect to the point Q is calculated:

$$\left(\frac{186}{29},\frac{25}{29},\frac{136}{29}\right)=\left(\frac{4+x}{2},\frac{2+y}{2},\frac{2+z}{2}\right) \Rightarrow S=\left(\frac{256}{29},-\frac{8}{29},\frac{214}{29}\right)$$

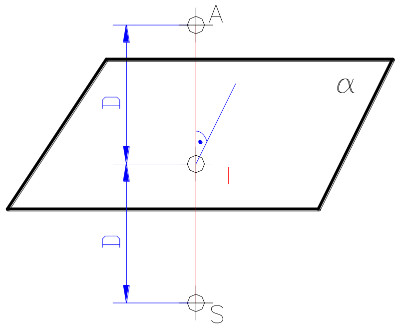
<u>5.3.G – Symmetric point with respect to a plane</u>

Given a point (A), its symmetric point (S) with respect to a plane (α) is located in a line (p) that passes through A and is perpendicular to the given plane. Assuming that the point of intersection between the line (p) and the plane is I, the symmetric point S is located in the line p a distance AI from the point I. The steps to obtain the symmetric point are the following:

1. Calculate the line p, which is perpendicular to the plane (α) and passes through the point (A)

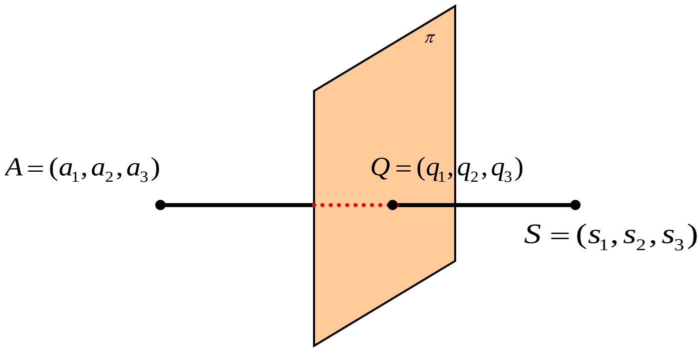
2. Calculate the point of intersection (I) between the line (p) and the plane (α)

3. Calculate the symmetric point (S) of the given point (A) with respect to the point I



5.3.A – Symmetric point with respect to a plane

Let $A=(a_1,a_2,a_3)$ and $S=(s_1,s_2,s_3)$ be two points. These points are symmetric with respect to the plane π, if they are the end points of the segment AS with bisector π.



The point Q, intersection point between the segment AS and the plane π, is the projection of the point A in the plane π. The points A and S are symmetric with respect to the point Q. The steps to obtain the symmetric point of A with respect to the plane π are the following:

1. Calculate the line r, which is perpendicular to the plane π and passes through the point A.

2. Calculate the point of intersection Q between the line r and the plane π.

3. Calculate the symmetric point S of the point A with respect to the point Q.

5.3. Mutual examples of both subjects

►***Example 38 (A)***

Determine the symmetric point of $A=(5,0,0)$ with respect to the plane $\alpha: 4x-y-4z=8$

Solution: We will calculate the line that passing through the point A is perpendicular to the given plane α. We will use the normal vector of the plane as the direction vector of the line: $\vec{a}=(4,-1,-4)$

$$r: \frac{x-5}{4} = \frac{y}{-1} = \frac{z}{-4} \Rightarrow \begin{cases} 5-x = 4y \\ 4y = z \end{cases}$$

Next, the point of intersection between the line r and the plane α will be calculated:

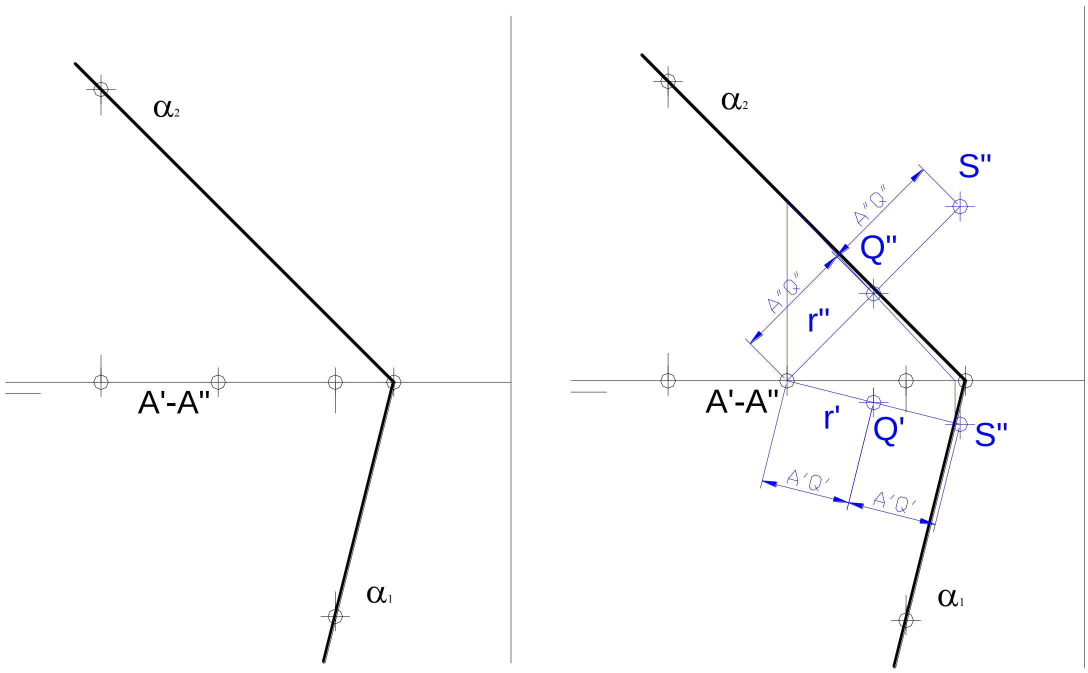
$$Q = \begin{cases} 5-x = 4y \\ 4y = z \\ 4x - y - 4z = 8 \end{cases} \Rightarrow Q = \left(\frac{39}{11}, \frac{4}{11}, \frac{16}{11}\right).$$

Finally the symmetric point of A with respect to the point Q is calculated:

$$\left(\frac{39}{11}, \frac{4}{11}, \frac{16}{11}\right) = \left(\frac{5+x}{2}, \frac{0+y}{2}, \frac{0+z}{2}\right) \Rightarrow S = \left(\frac{23}{11}, \frac{8}{11}, \frac{32}{11}\right)$$

►***Example 38 (G)***

Given a point A, determine its symmetric point with respect to the plane α.



Solution:

The line r that passing through the point A is perpendicular to the plane α is determined. We will use an auxiliary plane to calculate the intersection between the line r and the plane α. In the line r we take the distance AQ starting from the point Q and the symmetric point of A (the point S) is obtained.