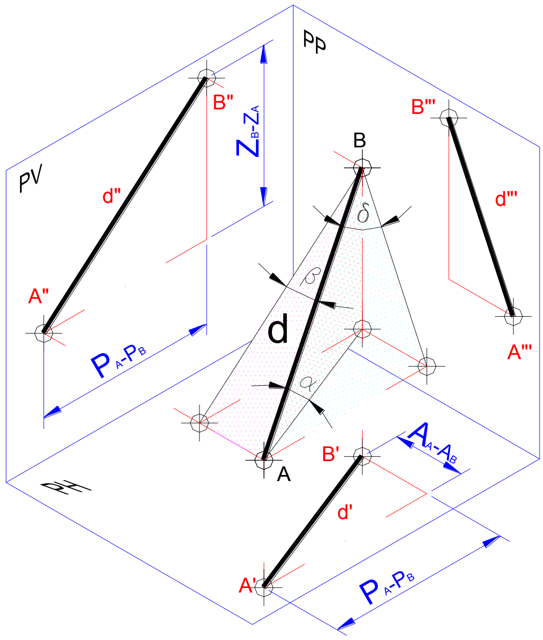
LESSON IV: DISTANCE BETWEEN ELEMENTS

4.1.G – Distance between two points

The real distance of the segment AB can be calculated using the metric relationships that the orthogonal projections of this segment in a plane satisfy. To do this some of the right triangles formed with the PH, PV or PP projections are built.



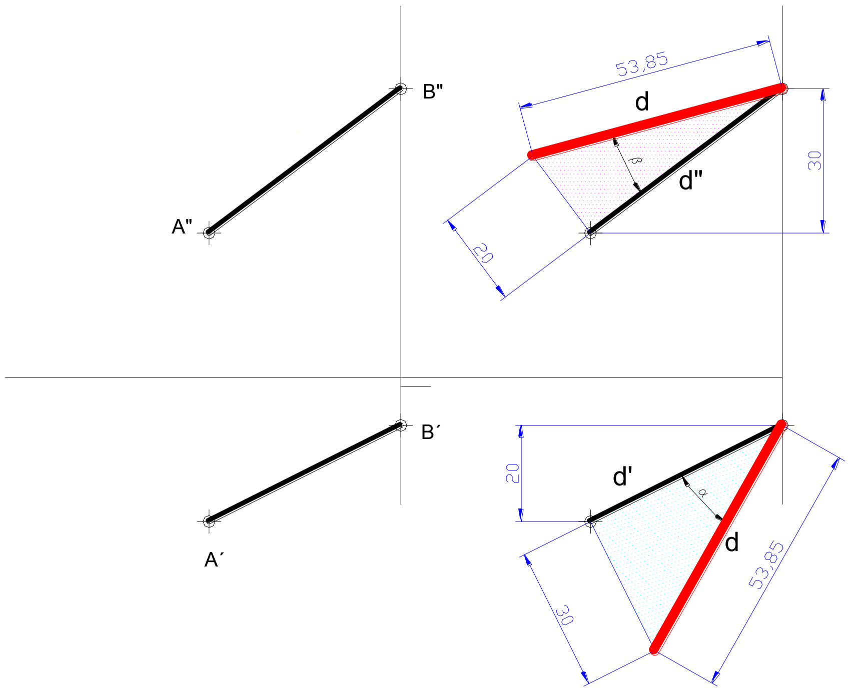
The angle between the segment and the projection planes PH, PV and PP can also be calculated by constructing these triangles. The angle between the segment and the projection plane PH is calculated using d'. The angle between the segment and the projection plane PV is calculated using d'' and the one formed with PP is calculated making use of d'''.

►*Example 26 (G)*

Calculate the distance between the points A(4,3,3) and B(0,1,6).

Solution:

The auxiliary triangles are built using the projections of the lines. It is enough to build only one of these triangles. In this example two triangles have been built, and it has been verified that the same result is obtained.



4.1.A – Distance between two points

Let A and B be two points in the space. The distance between these two points is the length of the segment $\overline{AB}$, which is the same as the modulus of the vector $\overrightarrow{AB}$. The distance between two points is denoted as $d(A,B)$ or $|\overrightarrow{AB}|$.

Let $A=(x_1,y_1,z_1)$ and $B=(x_2,y_2,z_2)$ be the coordinates of the points A and B. Then, the coordinates of the vector $\overrightarrow{AB}$ are given by:

$$\overrightarrow{AB}=(x_2-x_1,y_2-y_1,z_2-z_1)$$

And the analytic expression of the distance is given by:

$$d(A,B)=|\overrightarrow{AB}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

The distance between two points satisfies the following properties:

$$d(A,B)=0 \longleftrightarrow A=B$$

$d(A,B)=D(B,A)$ (Symmetry property)

$d(A,C)\leq d(A,B)+d(B,C)$ (Triangle inequality)

►***Example 27 (A)***

Calculate the distance between the points $A(4,3,3)$ and $B(4,3,6)$.

Solution: The distance between the given points is:

$$d(A,B)=\sqrt{(4-4)^2+(3-3)^2+(3-6)^2}=3$$

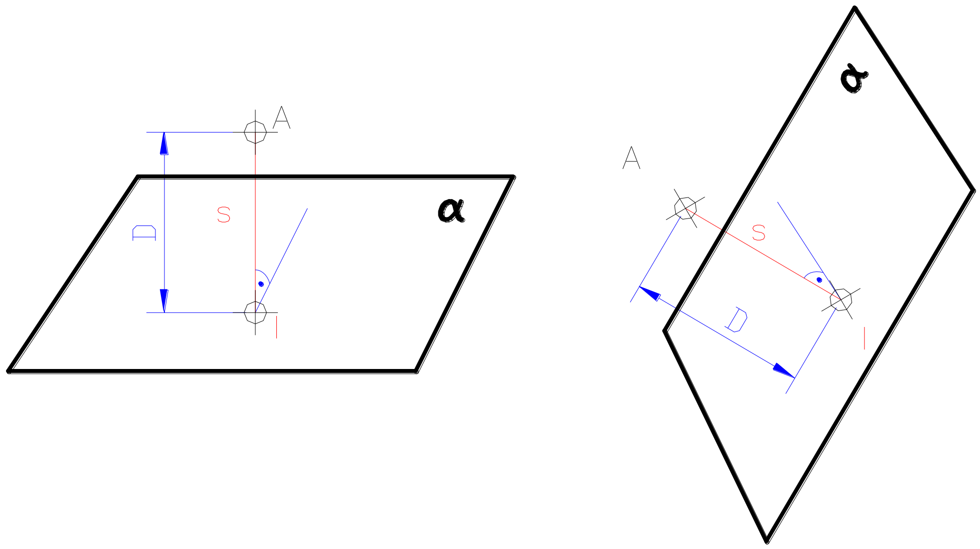
4.2.G – Distance between a point and a plane

The distance between a point and a plane is measured in the perpendicular line to the plane passing through the given point. The process is the following:

1.- The perpendicular line **s** to the plane passing through the point **A** is drawn.

2.- The point of intersection between the line **s** and the plane is calculated (point **I**).

3.- The searched solution is the real distance **AI**.



4.2.A – Distance between a point and a plane

Let P be a given point not in the plane α. If the point is included in the plane, the distance between them is zero. So, we will assume that the point is not in the plane. The distance between them is the length of the segment $\overline{PQ}$, being Q the orthogonal projection of the point P in the plane α.

Vector expression

Let P be a point and the plane α defined by $\alpha(A_\alpha, \vec{n}_\alpha)$, being A_α any point of the plane and $\vec{n}_\alpha$ the normal vector of the plane. Let Q be the orthogonal projection of P in the plane α.

The distance between the point P and the plane α is the modulus of the vector $\overrightarrow{QP}$:

$$d(P,\alpha) = |\overrightarrow{QP}|$$

In the right triangle $A_\alpha QP$ the following equality is satisfied: $\overrightarrow{A_\alpha P} = \overrightarrow{A_\alpha Q} + \overrightarrow{QP}$.

Doing the scalar product of both sides of the previous expression with the normal vector $\vec{n}_\alpha$, we get: $\overrightarrow{A_\alpha P} \cdot \vec{n}_\alpha = \overrightarrow{A_\alpha Q} \cdot \vec{n}_\alpha + \overrightarrow{QP} \cdot \vec{n}_\alpha = \overrightarrow{QP} \cdot \vec{n}_\alpha$

As $\overrightarrow{A_\alpha Q}$ and $\vec{n}_\alpha$ are perpendicular, the scalar product of them is zero.

By applying absolute values in the last expression we get:

$$|\overrightarrow{A_\alpha P}\cdot\vec{n}_\alpha| = |\overrightarrow{QP}\cdot\vec{n}_\alpha| = |\overrightarrow{QP}|\cdot|\vec{n}_\alpha|$$

Hence, the vector expression of the distance is obtained: $\boxed{d(P,\alpha) = |\overrightarrow{QP}| = \dfrac{|\overrightarrow{A_\alpha P}\cdot\vec{n}_\alpha|}{|\vec{n}_\alpha|}}$ (1)

Analytic expression:

Assume that the plane is determined by the equation in the general form: $\alpha:\ Ax+By+Cz+D=0$.

Let $A_\alpha(x_0,y_0,z_0)$ be a point included in the plane α, $\vec{n}_\alpha=(A,B,C)$ the normal vector of the plane and $P(x_1,y_1,z_1)$ a given point.

By substituting these values in expression (1):

$$\begin{aligned} d(P,\alpha) &= \frac{|(x_1-x_0,y_1-y_0,z_1-z_0)\cdot(A,B,C)|}{|(A,B,C)|} \\ &= \frac{|Ax_1+By_1+Cz_1-Ax_0-By_0-Cz_0|}{\sqrt{A^2+B^2+C^2}} \end{aligned}$$

On the other hand, as the point $A(x_0,y_0,z_0)$ is in the plane: $-Ax_0-By_0-Cz_0==D$. And we obtain the following expression: $\boxed{d(P,\alpha)=\dfrac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}}$

If the plane is not determined using the general form, the first step consists in obtaining its general form and following the described process.

To sum up, the distance between a point and a plane can be calculated by substituting the point in the equation of the plane and dividing this value by the modulus of the normal vector of the plane. If the result is negative, the absolute value is considered.

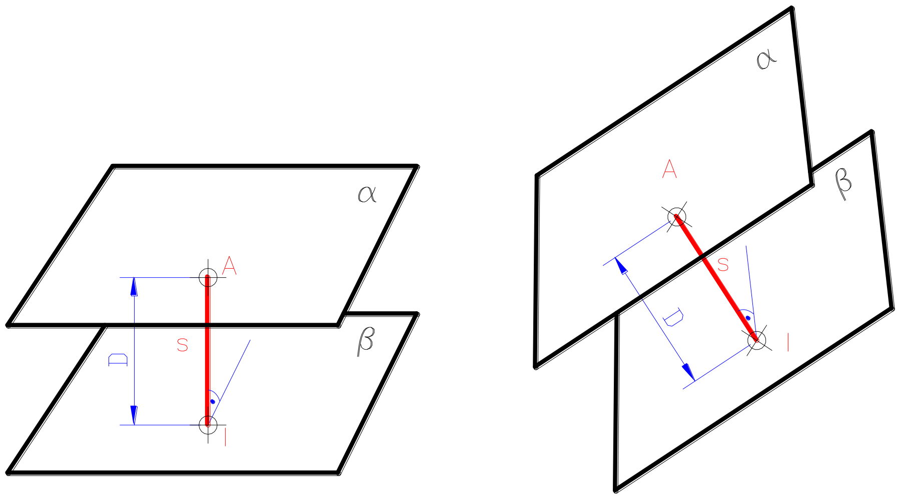
►***Example 28 (A)***

Calculate the distance between the point $A(1,0,1)$ and the plane $\alpha: 4x+y+4z=36$.

Solution: $$d(A,\alpha)=\frac{|4\cdot1+0\cdot1+4\cdot1-36|}{\sqrt{4^2+1^2+4^2}}=\frac{28}{\sqrt{33}}$$

4.3.G – Distance between two parallel planes

By considering any point (A) of one of the planes, we are in the case of the previous section. So, the distance between two parallel planes can be calculated by considering any point of one of the planes and by calculating the distance from this point to the other plane.



4.3.A – Distance between two parallel planes

Let α and β be two parallel planes. The distance between any point of one of the planes to the other plane can be calculated as:

$$d(\alpha,\beta) = d(P_\alpha,\beta) = d(P_\beta,\alpha)$$

To make calculations easier a point of the form $(0,0,z), (0,y,0)$ or $(x,0,0)$ can be considered. In this way x, y or z has to be calculated depending on the chosen case.

►*Example 29 (A)*

Calculate the distance between the plane $\alpha: 4x + y + 4z = 36$ and the plane β that passes through the points (2,0,0), (1,0,1) and (1,4,0).

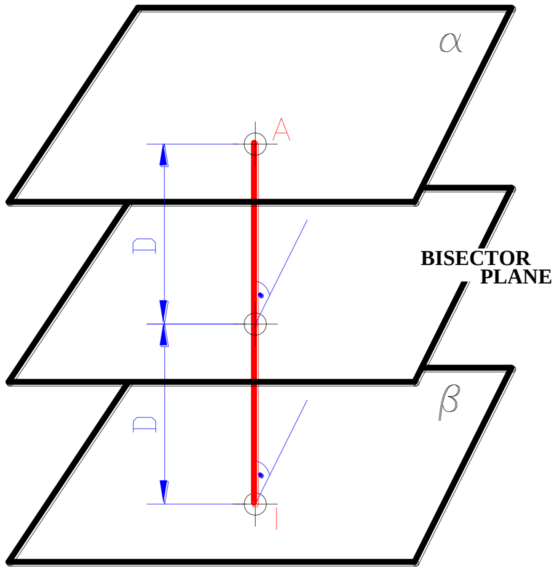
Solution: The general equation of the plane β is $\beta: 4x + y + 4z = 8$. On the other hand, as the normal vectors of the planes are proportional (in this case they are the same), the planes are parallel.

We will choose any point of the plane α. Let $P(9,0,0) \in \alpha$ be the chosen point. Next the distance between this point and the plane β has to be calculated:

$$d(P,\alpha) = \frac{|9 \cdot 4 + 0 \cdot 1 + 0 \cdot 4 - 8|}{\sqrt{4^2 + 1^2 + 4^2}} = \frac{28}{\sqrt{33}}$$

4.4.G – Bisector plane of two planes

It is the plane that being parallel to the given planes, passes through the middle point of the distance. It can be calculated in the same way as before.



4.4.A – Bisector plane of two planes

Given α and β two parallel planes, the bisector plane is the set of points equidistant from both planes. Given the equations of both planes:

$$\begin{aligned}\alpha &: A_1x+B_1y+C_1z+D_1=0\\ \beta &: A_2x+B_2y+C_2z+D_2=0\end{aligned},$$

as the planes are parallel, the normal vectors of the planes are proportional:

$\dfrac{A_1}{A_2}=\dfrac{B_1}{B_2}=\dfrac{C_1}{C_2}=\delta\in\mathbb{R}$, that is to say, $rank\begin{pmatrix}A_1 & B_1 & C_1\\ A_2 & B_2 & C_2\end{pmatrix}=1$

The bisector plane of two parallel planes is parallel to the given planes and equidistant from both planes. Hence, its general equation is given by: $\gamma: A_1x+B_1y+C_1z+M_1=0$.

►***Example 30 (A)***

Calculate the bisector plane of these two planes: $\begin{cases}\alpha: x-2y+3z-14=0\\ \beta: -x+2y-3z=0\end{cases}$.

Solution:

a) Consider any point of one of the planes and calculate the line *r* that passing through the chosen point is perpendicular to the planes:

If we consider $P_\beta=(0,0,0)\in\beta$, the parametric equations of *r* are determined by the normal vector $\vec{n}_\alpha$ or $\vec{n}_\beta$ and the chosen point: $r:\begin{cases}x &=0+t\\ y &=0-2t\\ z &=0+3t\end{cases}$

b) Next, we calculate the point of intersection P_α between the line *r* and the plane α:

As the line is given in parametric form, we substitute $x=t,\ y=-2t,\ z=3t$ in the equation of α : $(0+t)-2(0-2t)+3(0+3t)-14=0 \longleftrightarrow t=1$.

As a consequence, the point of intersection $P_\alpha=(1,-2,3)\in\alpha$ is obtained.

c) The midpoint P_γ of the segment $\overline{P_\alpha P_\beta}$ is calculated:

$$P_\gamma=\frac{P_\alpha+P_\beta}{2}=\frac{(1,-2,3)+(0,0,0)}{2}=\left(\frac{1}{2},-1,\frac{3}{2}\right)$$

d) Make the point P_γ belong to the plane γ:

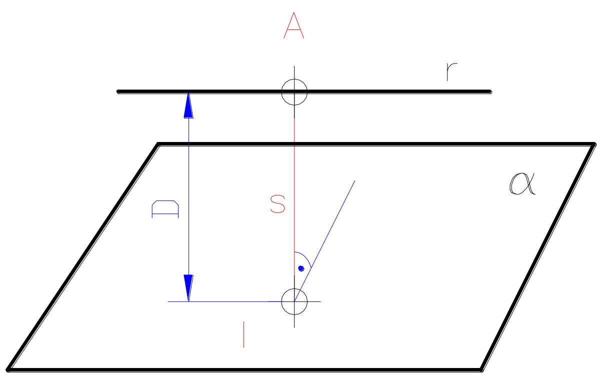
$$P_\gamma\in\gamma: x-2y+3z+M_1=0\rightarrow M_1=-7$$

Finally, the equation of the bisector plane of the planes α and β can be calculated:

$$\gamma: x-2y+3z-7=0$$

4.5.G – Distance between a line and a parallel plane

By choosing any point (A) of the line, the distance between the line and the plane can be calculated as the distance between the chosen point and the plane.



4.5.A - Distance between a line and a parallel plane

Let r and α be a line and a plane, defined respectively by $r(A_r,\vec{u}_r)$ and $\alpha(A_\alpha,\vec{n}_\alpha)$.

As r and α are parallel, $\vec{u}_r$, the direction vector of the line r, and $\vec{n}_\alpha$, the normal vector of the plane α, are perpendicular. That is to say, their scalar product is zero:

$\vec{u}_r \cdot \vec{n}_\alpha = 0$.

Geometrically, the distance between r and α can be calculated by choosing any point A_r in the line r and by calculating the distance from this point to the plane α: $d(r,\alpha) = d(A_r,\alpha)$.

Remark: If the resulting distance is zero, the line r is included in the plane α .

►*Example 31 (A)*

Calculate the distance between the line $r: \begin{cases} x = 3-3t \\ y = 6-t \\ z = 6t \end{cases}$ and the plane $\alpha: 6x+12y+5z-66=0$ which is parallel to the given line.

Solution: First of all, we will verify that the line *r* and the plane α are parallel:

$$\vec{u}_r \cdot \vec{n}_\alpha = (-3,-1,6)\cdot(6,12,5) = 0.$$

The distance between a line and a parallel plane can be calculated as the distance from any point of the line to the plane:

$$d((3,6,6),\alpha) = \frac{|6\cdot 3+12\cdot 6+5\cdot 0-66|}{\sqrt{6^2+12^2+5^2}} = \frac{24}{\sqrt{205}}$$

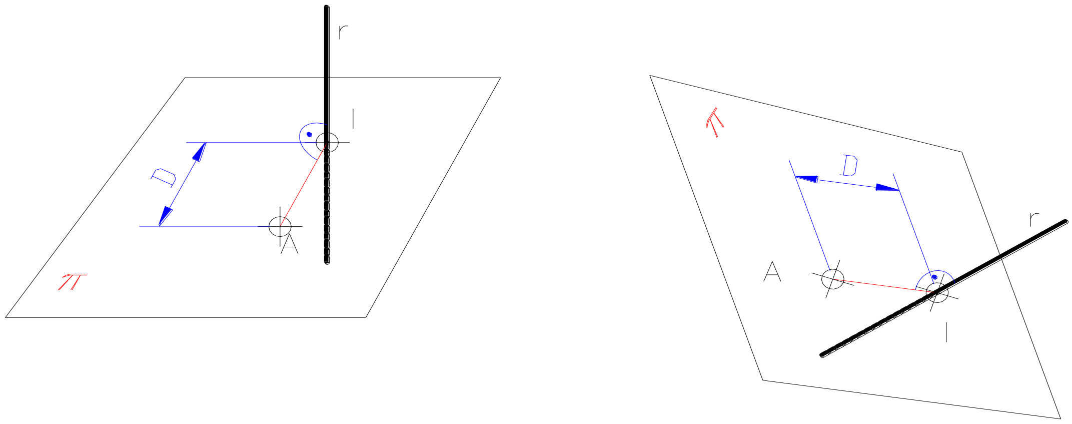
4.6.G – Distance from a point to a line

The distance from a point to a line is measured in a line that is perpendicular to the given line. That is to say, it is measured in a plane that is perpendicular to the given line.

1.- A perpendicular plane π to the line **r** passing through the point **A** is drawn.

2.- The point of intersection between the line **r** and the plane is calculated (point **I**).

3.- The searched solution is the real distance **AI**.



4.6.A – Distance from a point to a line

Let P be a given point not in the line. If the point is included in the line, the distance between them is zero. So, we will assume that the point is not in the line. The distance between them is the length of the segment $\overline{PQ}$, being Q the orthogonal projection of the point P in the line.

Vector expression

Let P be a point and the line r defined by $r(A_r, \vec{u}_r)$, being A_r any point of the line and $\vec{u}_r$ the direction vector of the line. Let Q be the orthogonal projection of P in the line r.

The distance between the point P and the line r is the modulus of the vector $\overrightarrow{QP}$:

$$d(P,r) = |\overrightarrow{QP}|$$

In the right triangle A_rQP the following equality is satisfied: $\overrightarrow{A_rP} = \overrightarrow{A_rQ} + \overrightarrow{QP}$.

Doing the vector product of both sides of the previous expression with the direction vector $\vec{u}_r$, we get: $\overrightarrow{A_rP} \times \vec{u}_r = \overrightarrow{A_rQ} \times \vec{u}_r + \overrightarrow{QP} \times \vec{u}_r = \overrightarrow{QP} \times \vec{u}_r$

As $\overrightarrow{A_rQ}$ and $\vec{u}_r$ are parallel, the vector product of them is zero.

By applying absolute values in the last expression we get:

$$|\overrightarrow{A_rP} \times \vec{u}_r| = |\overrightarrow{QP} \times \vec{u}_r| = |\overrightarrow{QP}| \cdot |\vec{u}_r|$$

Hence, $d(P,r) = |\overrightarrow{QP}| = \dfrac{|\overrightarrow{A_rP} \times \vec{u}_r|}{|\vec{u}_r|}$ (2)

The distance from a point to a line can be calculated by dividing by the base (of modulus $|\vec{u}_r|$) the area of the parallelogram built with the vectors $\vec{u}_r$ and $\overrightarrow{A_rP}$

Analytic expression

Assume that the line r is determined by its continuous equation: $r: \dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$.

Let $A_r(x_0, y_0, z_0)$ be a point of the line r and $\vec{u}_r = (a,b,c)$ its direction vector. Let $P(x_1, y_1, z_1)$ be a point not included in the line. By substituting these values in expression (2):

$$d(P,r) = \frac{|(x_1 - x_0, y_1 - y_0, z_1 - z_0) \times (a,b,c)|}{|(a,b,c)|}$$

If the line is not determined using its continuous equation, the first step consists in obtaining a point of the line and its direction vector, and following the described process afterwards.

►***Example 32 (A)***

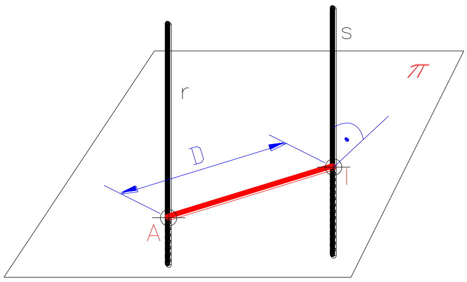
Calculate the distance from the point $A(7,1,5)$ to the line $r:\dfrac{x-1}{4}=\dfrac{y}{2}=\dfrac{z-5}{-3}$.

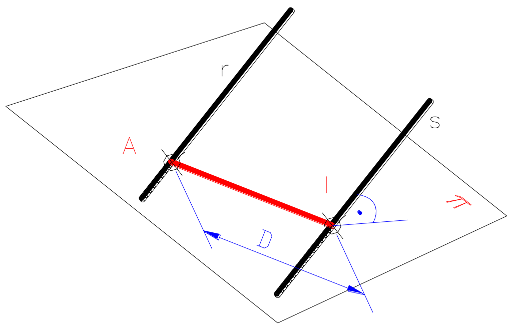
Solution: Any point of the line *r* is considered. We will take the point P(1,0,5). Next we apply the formula to calculate the distance from a point to a line, being $\vec{u}_r=(4,2,-3)$, $\overrightarrow{AP}\times\vec{u}_r=(3,-18,-8)$ and $|\vec{u}_r|=\sqrt{29}$.

$$d(A,r)=\frac{|\overrightarrow{AP}\times\vec{u}_r|}{|\vec{u}_r|}=\frac{\sqrt{397}}{\sqrt{29}}$$

4.7.G –Distance between two parallel lines

By choosing any point (A) of one of the lines, the distance between two parallel lines can be calculated as the distance between the chosen point and the other line.





4.7.A – Distance between two parallel lines

The distance between two parallel lines is the distance from any point of one of the lines to the other line: $d(r,s)=d(P_r,s)=d(P_s,r)$.

After considering the points P_r or P_s, the distance from a point to a line is calculated.

►Example 33 (A)

Calculate the distance between the lines $r: \frac{x-1}{4} = \frac{y}{2} = \frac{z-5}{-3}$ and $s: \begin{cases} x = 7+4t \\ y = 1+2t \\ z = 5-3t \end{cases}$.

Solution:

Both lines are parallel because their direction vectors are proportional. Considering the point $A(7,1,5) \in s$, we are in the case in which the distance from a point to a line has to be calculated. Hence the searched distance is: $d(r,s) = d(r,A) = \sqrt{\frac{397}{29}}$

4.8.G – Distance between two skew lines

Two skew lines are not parallel and do not intersect. The distance between two skew lines is measured in a line that is orthogonal to the given lines. If we only want to calculate the distance between the lines the process is simpler than in the case in which we want to calculate the line that intersects the given lines.

The process is the following:

1.- Any point **A** of one of the lines is chosen, and the line **r1** that passing through this point is parallel to the other line is built. These two lines define the plane α.

2.- Any point **B** of the other line is chosen, and the line **p1** that passing through this point is perpendicular to the plane α is drawn.

3.- The point of intersection **C** between the line **p1** and the plane is calculated.

4.- The searched solution is the real distance **BC**.

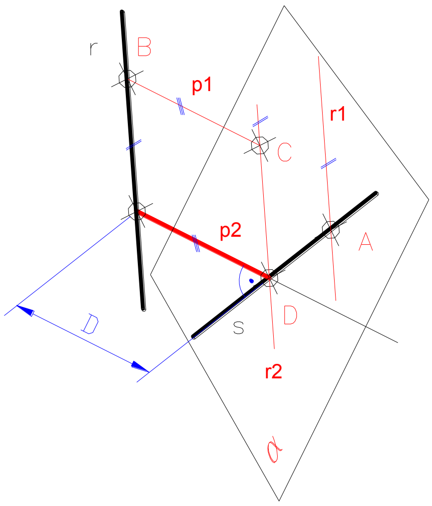
If we want the segment to be supported by the lines, the process continues as follows:

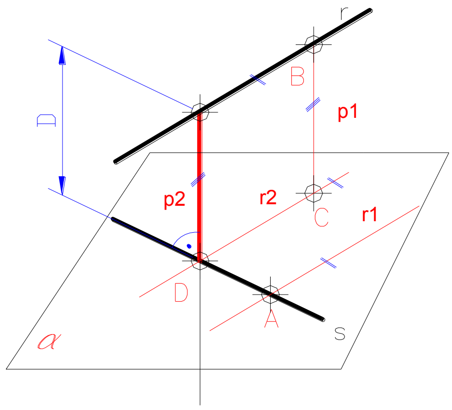
5.- The line **r2** that passing through the point **C** is parallel to the line **r** is drawn.

6.- **r2** and **s** are on the same plane and they are not parallel, so they intersect in a point **D**.

7.- Draw the line **p2** that passing through the point **D** is parallel to the line **p1**. As the lines **p2** and **r** are in the same plane, they intersect in a point (point **I**).

8.-The searched solution is **DI**.





4.8.A – Distance between two skew lines

The distance between two skew lines r and s is the distance between the plane that passing through the line r is parallel to the line s and the plane that passing through the line s is parallel to the line r.

Vector expression

Let *r* and *s* be two lines defined by $r(A_r,\vec{u}_r)$ and $s(A_s,\vec{u}_s)$. The distance between the lines *r* and *s*, is the distance from the point A_s to the plane $\alpha(A_r,\vec{u}_r,\vec{u}_s)$: $d(r,s)=d(A_s,\alpha)$

The vector expression of the distance from the point *P* to the plane α is given by:

$$d(P,\alpha)=\frac{|\overrightarrow{A_\alpha P}\cdot\vec{n}_\alpha|}{|\vec{n}_\alpha|}$$

In our case, by considering $A_\alpha=A_r$, $P=A_s$ and $\vec{n}_\alpha=\vec{u}_r\times\vec{u}_s$ we get the following expression: $d(r,s)=d(A_s,\alpha)=)=\dfrac{|\overrightarrow{A_rA_s}\cdot(\vec{u}_r\times\vec{u}_s)|}{|\vec{u}_r\times\vec{u}_s|}$

Using the expression of the mixed product we obtain:

$$\boxed{d(r,s)=\frac{|det(\overrightarrow{A_rA_s},\vec{u}_r,\vec{u}_s)|}{|\vec{u}_r\times\vec{u}_s|}} \quad (3)$$

Analytic expression:

Let assume that the continuous equations of the lines are:

$$r:\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \qquad s:\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$$

The vectors that will be used in expression (3) are:

$\vec{u}_r = (a_1, b_1, c_1)$, $\vec{u}_s = (a_2, b_2, c_2)$, $\overrightarrow{A_r A_s} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

As the formula obtained using these values is complicated, in practice expression (3) is applied using the numerical values of the problem.

If the lines are not given using their continuous form, it is necessary to obtain a point and the direction vector of each of the lines. The expression (3) is applied afterwards.

Remark: In some cases it is easier to develop the definition of the distance between two lines. In that case it is enough to consider the point A_s and to calculate the plane α that passing through the point $A_r(x_1, y_1, z_1)$ contains the line *r* and is parallel to the line *s*. Next, we calculate the distance from the point A_s to the plane α. The equation of the cited plane α can be calculated using the determinant:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

►***Example 34 (A)***

Calculate the distance between the lines $r: \dfrac{x-13}{4} = \dfrac{y}{5} = \dfrac{z-5}{-3}$ and $s: \begin{cases} x = 3 + 5t \\ y = 2 + t \\ z = 4t \end{cases}$.

Solution:

In this example, we will apply the analytic formula to calculate the distance between two skew lines.

The line *r* is determined by the point $B(13,0,5)$ and the direction vector $\vec{u}_r = (4,5,-3)$. And the line *s* is determined by the point $A(3,2,0)$ and the direction vector $\vec{u}_s = (5,1,4)$ The distance between both lines is:

$$d(r,s) = \frac{|\,det(\overrightarrow{AB}, \vec{u}_r, \vec{u}_s)\,|}{|\,\vec{u}_r \times \vec{u}_s\,|} = \frac{187}{\sqrt{1931}}$$

being:

$\overrightarrow{AB} = B - A = (10,-2,5)$, $\quad det(\overrightarrow{AB}, \vec{u}_r, \vec{u}_s) = \begin{vmatrix} 10 & -2 & 5 \\ 4 & 5 & -3 \\ 5 & 1 & 4 \end{vmatrix} = 187$,

$|\,\vec{u}_r \times \vec{u}_s\,| = |\,(23,-31,-21)\,| = \sqrt{1931}$

Mutual perpendicular of two lines

The mutual perpendicular of two skew lines is the line that intersects perpendicularly each of the given lines. The mutual perpendicular p of two lines r and s is determined by the intersection of the panes $\alpha(A_r, \vec{u}_r, \vec{u}_r \times \vec{u}_s)$ and $\beta(A_s, \vec{u}_s, \vec{u}_r \times \vec{u}_s)$.

As a result, the analytic expression of the mutual perpendicular is:

$p: \begin{cases} det(\overrightarrow{A_r X}, \vec{u}_r, \vec{u}_r \times \vec{u}_s) & = 0 \\ det(\overrightarrow{A_s X}, \vec{u}_s, \vec{u}_r \times \vec{u}_s) & = 0 \end{cases}$, being X any point of the mutual perpendicular p.

The distance between two skew lines is the distance between the points of intersection of the mutual perpendicular with each of the given lines.

Another way to obtain the mutual perpendicular is by using the technique of generic points:

Let P_r and P_s be the intersection points of the mutual perpendicular with the skew lines r and s. The coordinates of the point P_r will be the parametric equations of the line r, that is to say: $P_r(x_1 + a_1 t, y_1 + b_1 t, z_1 + c_1 t)$. Similarly, the coordinates of the point P_s will be the parametric coordinates of the line s: $P_s(x_2 + a_2 s, y_2 + b_2 s, z_2 + c_2 s)$

As the vector $\overrightarrow{P_r P_s}$ is perpendicular to $\vec{u}_r$ and $\vec{u}_s$, their scalar products are zero:
$\begin{cases} \overrightarrow{P_r P_s} \cdot \vec{u}_r & = 0 \\ \overrightarrow{P_r P_s} \cdot \vec{u}_s & = 0 \end{cases}$

By doing the calculations, a system of two equations and two unknowns (*t* and *s*) is obtained.

After solving the system, the values of the parameters *t* and *s* are substituted in P_r and P_s, obtaining the coordinates of these two points.

Once the coordinates of the points P_r and P_s are known, the equation of the mutual perpendicular p can be calculated. The direction vector of the mutual perpendicular is $\overrightarrow{P_r P_s}$. The distance between the lines *r* and *s* is given by the modulus $|\overrightarrow{P_r P_s}|$.

►***Example 35 (A)***

Obtain the equation of the mutual perpendicular of the lines $r: x = y = z$ and $s: x = y = 3z - 1$.

Solution: Using the equations of the given lines we conclude:
$A_r(0,0,0), \vec{u}_r = (1,1,1), A_s(-1,-1,0), \vec{u}_s = (3,3,1) \rightarrow \vec{u}_r \times \vec{u}_s = (-2,2,0)$

The mutual perpendicular p is determined by the intersection of the planes:

$$det(\overrightarrow{A_r X},\vec{u}_r,\vec{u}_r\times\vec{u}_s)=0\rightarrow\begin{vmatrix} x & y & z\\ 1 & 1 & 1\\ -2 & 2 & 0\end{vmatrix}=0\leftrightarrow x+y-2z=0$$

$$det(\overrightarrow{A_s X},\vec{u}_s,\vec{u}_r\times\vec{u}_s)=0\rightarrow\begin{vmatrix} x+1 & y+1 & z\\ 3 & 3 & 1\\ -2 & 2 & 0\end{vmatrix}=0\leftrightarrow x+y-6z+2=0$$

As a consequence, the equation of the mutual perpendicular is: $p:\begin{cases} x+y-2z & =0\\ x+y-6z+2 & =0\end{cases}$.

Using the technique of generic points of each of the lines, we have:

$P_r=(t,t,t),\ P_s=(3s-1,3s-1,s)\quad \overrightarrow{P_rP_s}=(3s-t-1,3s-t-1,s-t)$

This vector must be perpendicular to the direction vectors $\vec{u}_r$ and $\vec{u}_s$:

$\overrightarrow{P_rP_s}\cdot\vec{u}_r=(3s-t-1,3s-t-1,s-t)\cdot(1,1,1)=0$

$\overrightarrow{P_rP_s}\cdot\vec{u}_s=(3s-t-1,3s-t-1,s-t)\cdot(3,3,1)=0$

Doing some calculations we get: $\begin{cases} 7s-3t & =2\\ 19s-7t & =6\end{cases}\leftrightarrow\begin{cases} t & =\dfrac{1}{2}\\ s & =\dfrac{1}{2}\end{cases}$

The points that correspond to the values of the parameters *t* and *s* are $P_r=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ and $P_s=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$.

Both points are the same, which means that the lines intersect each other, being the distance between them zero.

The mutual perpendicular passes through the point $P_r=(1/2,1/2,1/2)$ and its direction vector is $\vec{u}_r\times\vec{u}_s=(-2,2,0)$. So the equation of the mutual perpendicular is:

$$p:\frac{x-\frac{1}{2}}{-2}=\frac{y-\frac{1}{2}}{2}=\frac{z-\frac{1}{2}}{0}\quad\Rightarrow\quad p:\begin{cases} x+y-1=0\\ z=\dfrac{1}{2}\end{cases}$$