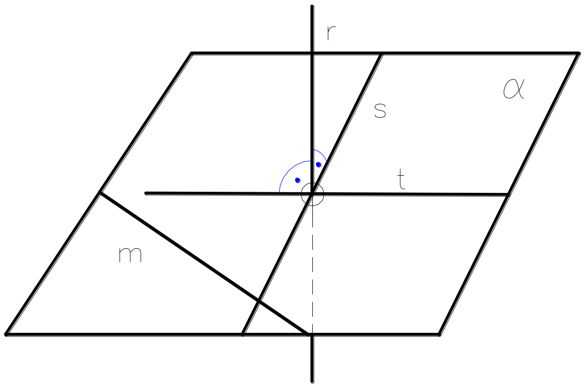
# LESSON III: ORTHOGONALITY

## 3.1.G – Perpendicular lines and planes

A line is perpendicular or orthogonal to a plane if it is perpendicular to two non-parallel lines passing from the foot of the line.

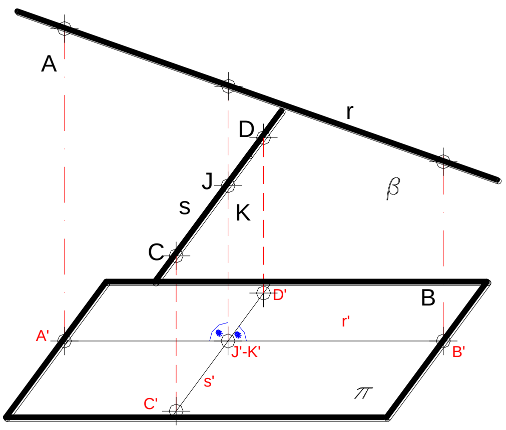


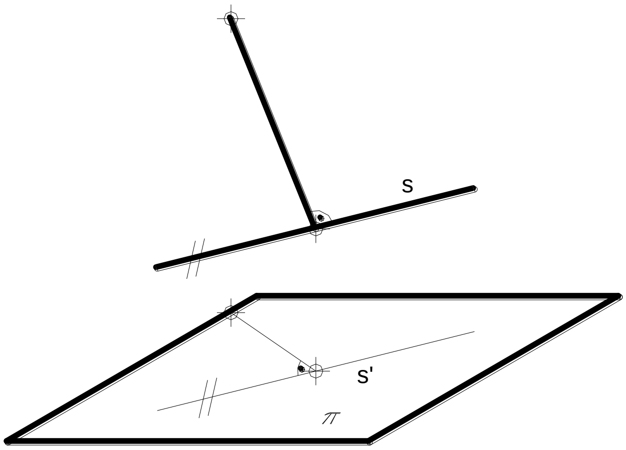

From the previous statement it can be concluded that in order to check the orthogonality of a line and a plane, it is enough if the line is perpendicular to two non-parallel lines in the plane or to two non-parallel lines that are parallel to the plane. If a line is perpendicular to a plane, it is perpendicular to all lines in the plane.

If two lines are parallel, every plane which is perpendicular to one of the lines will be perpendicular to the other line. If two planes are parallel, any line which is perpendicular to one of the planes will be perpendicular to the other plane. Hence, two planes that are perpendicular to a line are parallel and two lines that are perpendicular to a plane are parallel.

If a line is perpendicular to a plane, any line that is perpendicular to this line is parallel to the plane or it is included in it.

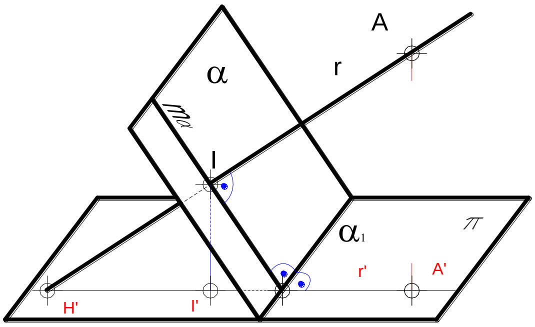
The theorem of the three perpendiculars is an important theorem to analyse the orthogonality of projections. This theorem says the following: If two lines are perpendicular in the space being one of them parallel to the plane of projection, the lines are projected perpendicularly on this plane.

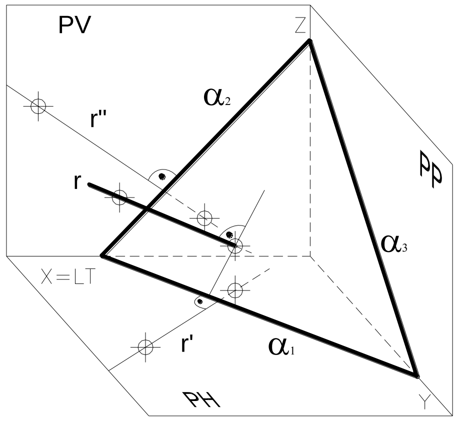


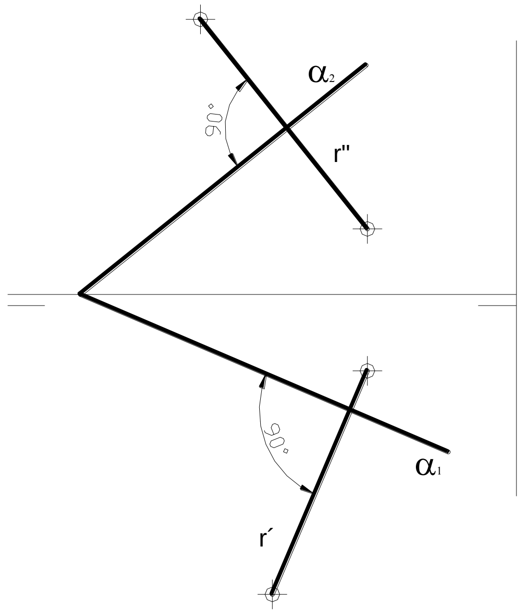
## COROLLARY

*If a line "r" is perpendicular to a plane **α**, its projections are perpendicular to the homonymous traces of the plane.*




As it has been said before, if a line is perpendicular to a plane, the projections of the line are perpendicular to the homonymous traces of the plane: **r'** is perpendicular to **$\alpha_1$** and **r''** is perpendicular to **$\alpha_2$**.




## 3.1.A – Perpendicular lines and planes

Let be *r* a line with direction vector $\vec{u} = (u_1, u_2, u_3)$ and $\pi : Ax + By + Cz + D = 0$ a plane with normal vector $\vec{a} = (A, B, C)$. The line and the plane are perpendicular if the direction vector of the line and the normal vector of the plane are parallel. That is to say, the angle formed by these two vectors is 0º. This means that the line *r* and the plane $\pi$ are perpendicular if the sine of the angle formed by them is 0, $\sin\theta = 0$. This means that the vectors $\vec{u}$ and $\vec{a}$ are linearly dependent $\Rightarrow \dfrac{A}{u_1} = \dfrac{B}{u_2} = \dfrac{C}{u_3}$

## 3.1. Mutual examples of both subjects

**►*Example 17 (A)***

Find the line that passing through the point $P(7,3,4)$ is perpendicular to the plane that passes through the points (8,0,0), (3,0,2) and (5,4,0). Obtain the intersection point between the line and the plane.

*Solution*: The implicit equation of the plane $\alpha$ is the following:

$$\alpha : \begin{vmatrix} x-8 & y & z \\ 3-8 & 0-0 & 2-0 \\ 5-8 & 4-0 & 0-0 \end{vmatrix} = 0 \mapsto \quad \alpha : 4x+3y+10z=32$$

The equation of the line p can be calculated as its direction vector $\vec{n}_\alpha = (4,3,10) = \vec{u}_p$ and one of its points are known $P(7,3,4)$:

$$p: \frac{x-7}{4} = \frac{y-3}{3} = \frac{z-4}{10}$$

The intersection point A between the line $p$ and the plane $\alpha$ can be calculated by calculating the point that belonging to the line p satisfies the equation of the plane $\alpha$. That is to say, among all the points of the line $p$ ($x=4t+7, y=3t+3, z=10t+4$) there is one included in the plane:

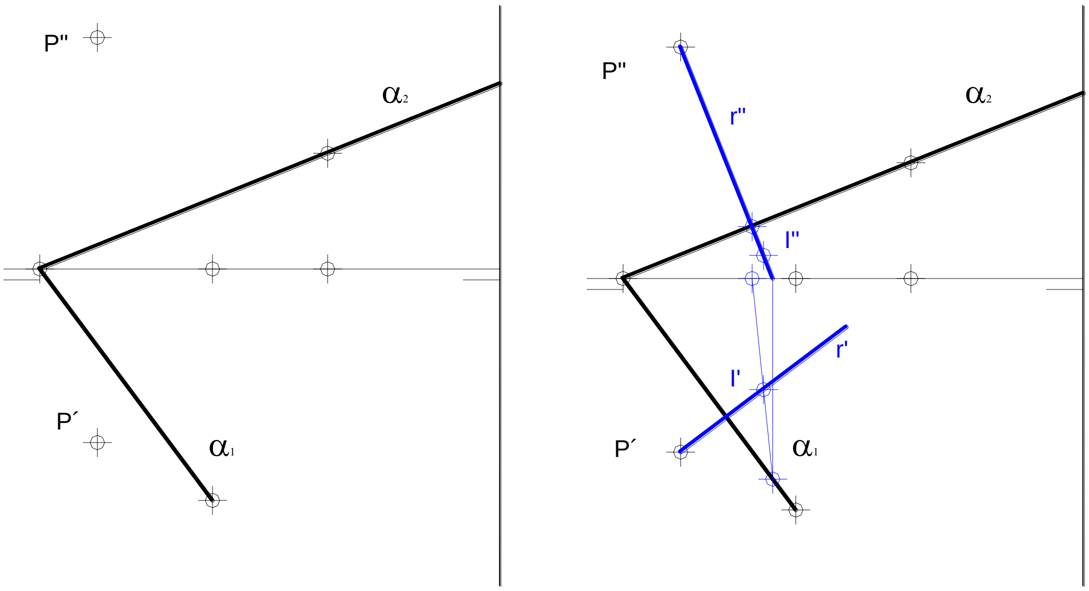
$$4(4t+7)+3(3t+3)+10(10t+4)=32 \mapsto t=-9/25$$

The intersection point is given by: $A\left(\frac{139}{5}, \frac{48}{5}, \frac{2}{5}\right)$

**►*Example 18 (G)***

Draw a line that passing through the point P is perpendicular to the plane $\alpha$. Obtain the point of intersection.

*Solution*: The vertical projection of the solution line r is perpendicular to the plane $\alpha_2$ and its horizontal projection is perpendicular to $\alpha_1$. The point of intersection is calculated by using the procedures explained in the previous section.




### ►*Example 19 (A)*

Obtain a plane $\alpha$ that being perpendicular to the line $r\,((9,0,2),(7,4,5))$, passes through the point $P(11,2,4)$. Obtain the intersection point of them.

*Solution:* The continuous equation of the line is:

$$r: \frac{x-9}{-2} = \frac{y}{4} = \frac{z-2}{3}$$

The normal vector of the plane $\alpha$ is $\overrightarrow{n_\alpha} = \overrightarrow{u_r} = (-2,4,3)$ and the plane passes through $P = (11,2,4)$.

$$\alpha : -2(x-11) + 4(y-2) + 3(z-4) = 0 \mapsto -2x + 4y + 3z + 2 = 0$$

Next, the point of intersection A between the line $r$ and the plane $\alpha$ can be calculated.

Among all the points included in the line $(x = -2t + 9, y = 4t, z = 3t + 2)$, a point that satisfies the equation of the plane $\alpha$ is calculated, which corresponds to the value $t = 10/29$. Therefore, the point of intersection of the line and the plane is:

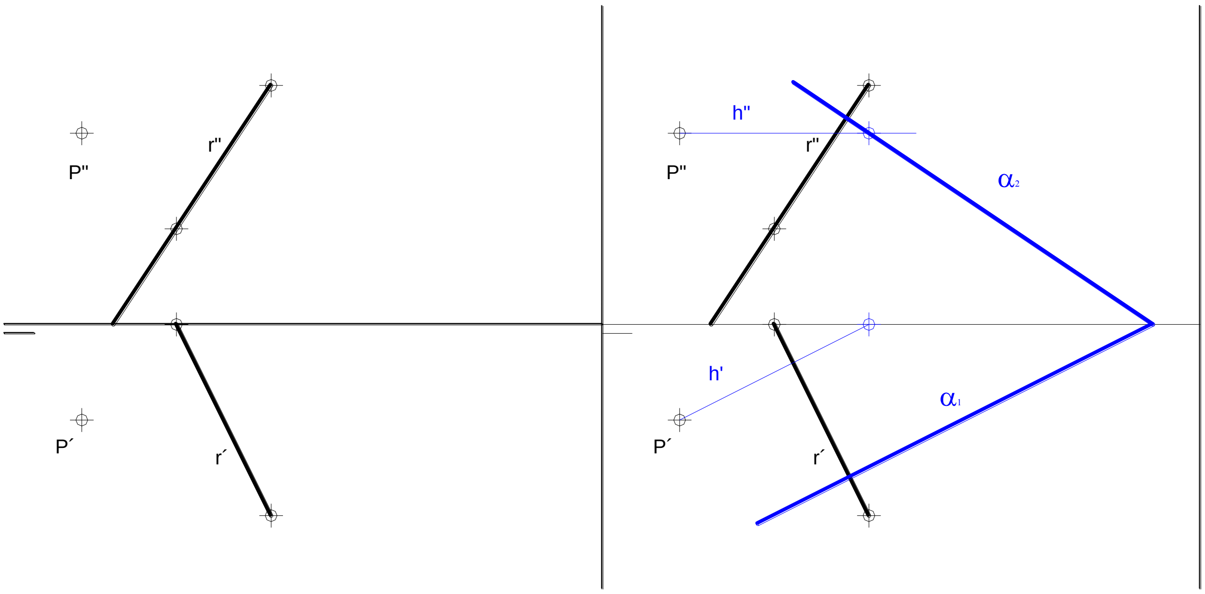
$$A\left(\frac{241}{29}, \frac{40}{29}, \frac{88}{29}\right)$$

### ►*Example 20 (G)*

Draw a plane that passing through the point **P** is perpendicular to the line **r**.

*Solution*: The projections of the line are perpendicular to the homonymous traces of the plane: **r'** is perpendicular to $\boldsymbol{\alpha_1}$ and **r''** is perpendicular to $\boldsymbol{\alpha_2}$.

The traces of the plane are calculated by drawing the parallels to the horizontal and vertical traces: the horizontal **h** and the frontal **f**. In this case, a horizontal to the plane has been drawn and the traces have been calculated using this line.



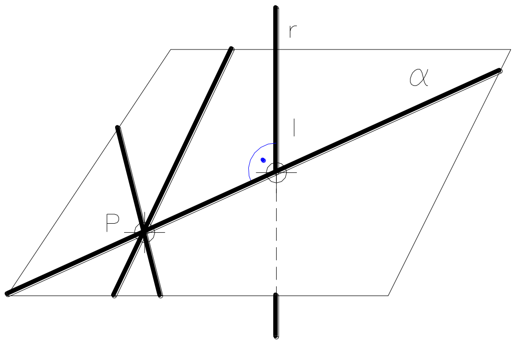

## 3.2.G – Perpendicular lines

All the perpendicular lines to a given line are included in a plane that is perpendicular to the given line.

►***Example 21 (G)***

| |
|---|
| Draw a line that passing through the point **P** is perpendicular to the line **r** and cuts it. |

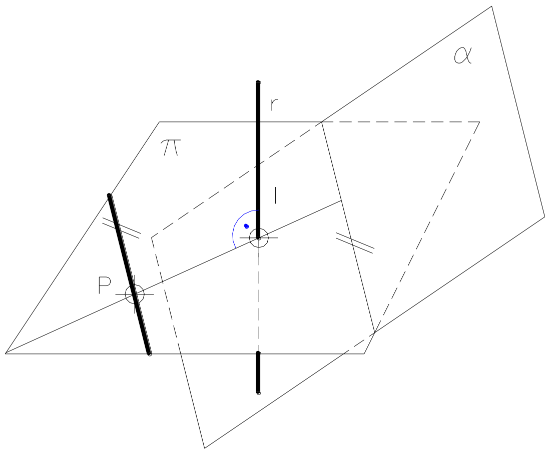
*Solution*: A plane that containing the point **P** is perpendicular to the line **r** is drawn. The solution passes through the point of intersection of the line and the plane, and the point **P**.




►***Example 22 (G)***

| |
|---|
| Draw a line that passing through the point **P** is perpendicular to the line **r** and parallel to the plane α. |

*Solution*: A plane that containing the point **P** is perpendicular to the line **r** is drawn. Next, the intersection of both planes is calculated. The solution line contains the point **P** and it is parallel to the line of intersection of both planes.




## 3.2.A – Perpendicular lines

Let *r* and *s* be two lines with direction vectors $\vec{u}=(u_1,u_2,u_3)$ and $\vec{v}=(v_1,v_2,v_3)$ respectively. These two lines are perpendicular if their direction vectors are perpendicular. That is to say, the angle formed by their direction vectors is 90º. Hence, the cosine of this angle is 0: $\cos\theta=0 \Rightarrow u_1v_1+u_2v_2+u_3v_3=0$

►***Example 23 (A)***

Calculate a line that containing the point $P(12,3,6)$ is perpendicular to the line $r:\dfrac{x-6}{-6}=\dfrac{y-5}{3}=\dfrac{z-4}{3}$ and cuts it. Calculate the point of intersection of both lines.

*Solution:* We define the plane $\alpha$ that contains the point $P$ and has the direction vector of the line $r$ as its normal vector:

$$\alpha:-6(x-12)+3(y-3)+3(z-6)=0 \mapsto \alpha:-2x+y+z+15=0$$

We calculate the point of intersection $Q$ between the plane $\alpha$ and the line $r$:

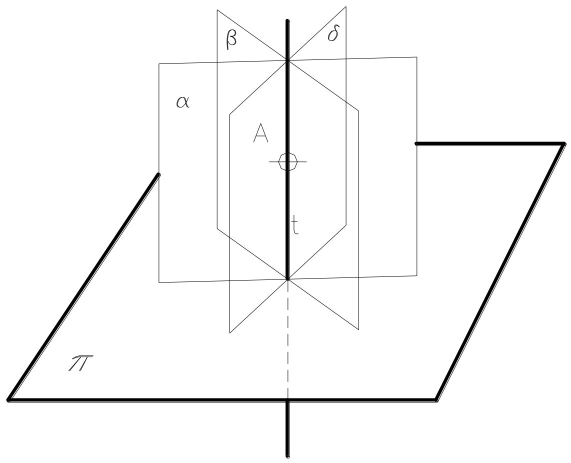
$-2(6-6t)+1(5+3t)+1(4+3t)+15=0 \ \rightarrow \ t=-2/3$, hence, $Q=(10,3,2)$.

The perpendicular line that we want to find passes through the points $P$ and $Q$:

$$p:\begin{cases} y=3 \\ 2x-z=18 \end{cases}$$

## 3.3.G – Perpendicular planes

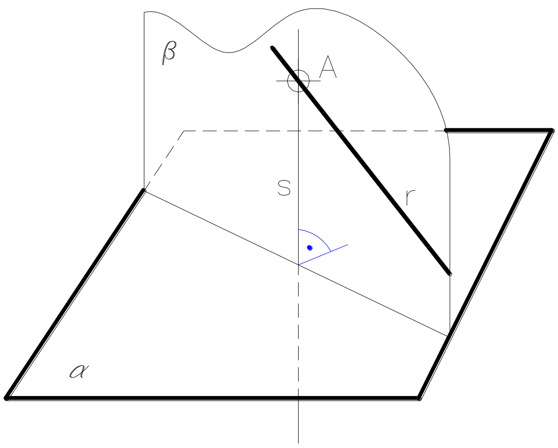
A plane **B** is perpendicular to another plane **A**, if **B** contains a line that is perpendicular to plane **A**. So given a plane, in order to obtain a perpendicular plane to the given plane we have to draw a perpendicular line to the given plane. Any plane of the sheaf of planes that contain this line is perpendicular to the given plane.




►***Example 24 (G)***

Draw a plane that containing the line **r** is perpendicular to the plane **α**.

*Solution*: From a point of the line **r**, a line **s** that is perpendicular to the plane **α** has to be drawn. These two lines **r** and **s** define the solution plane.




## 3.3.A – Perpendicular planes

Let $\pi_1 : A_1x + B_1y + C_1z + D_1 = 0$ be a plane with normal vector $\overrightarrow{a_1} = (A_1, B_1, C_1)$ and $\pi_2 : A_2x + B_2y + C_2z + D_2 = 0$ a plane with normal vector $\overrightarrow{a_2} = (A_2, B_2, C_2)$. These two planes are perpendicular if their normal vectors are perpendicular. That is to say, the angle formed by its normal vectors is 90º, which means that the cosine of this angle is zero:

$$\cos\theta = 0 \Rightarrow A_1A_2 + B_1B_2 + C_1C_2 = 0$$

### ►*Example 25 (A)*

Draw from the point $P(4,3,1)$ planes which are perpendicular to the plane $\alpha: 5x+6y+4z=50$ and parallel to the line *r* defined by the points $(6,5,4)$ and $(0,8,7)$.

*Solution:* Planes $\beta$ which are perpendicular to $\alpha$ and contain the point $P(4,3,1)$ satisfy the equation: $\beta: A(x-4)+B(y-3)+C(z-1)=0$, being the scalar product of the normal vectors zero:

$$(A,B,C)\cdot(5,6,4)=0 \mapsto A=\frac{-6B-4C}{5}$$

Hence the general equation of the plane $\beta$ is:

$$\beta: \frac{-6B-4C}{5}(x-4)+B(y-3)+C(z-1)=0, \forall\, B,C \in \mathbb{R}$$

The direction vector of the line $r$ is $\overrightarrow{v_r}=(-2,1,1)$. As the plane $\beta$ is parallel to the line $r$, vectors $\overrightarrow{n_\beta}$ and $\overrightarrow{v_r}$ are perpendicular: $\overrightarrow{n_\beta}.\overrightarrow{v_r}=0$. Hence, $B=\frac{-13C}{17}$ .

By substituting the values of $B$ and $A$ in the expression of the plane $\beta$ we get:

$$\beta: 2x-13y+17z+14=0$$