LESSON II: RELATIVE POSITIONS AMONG ELEMENTS

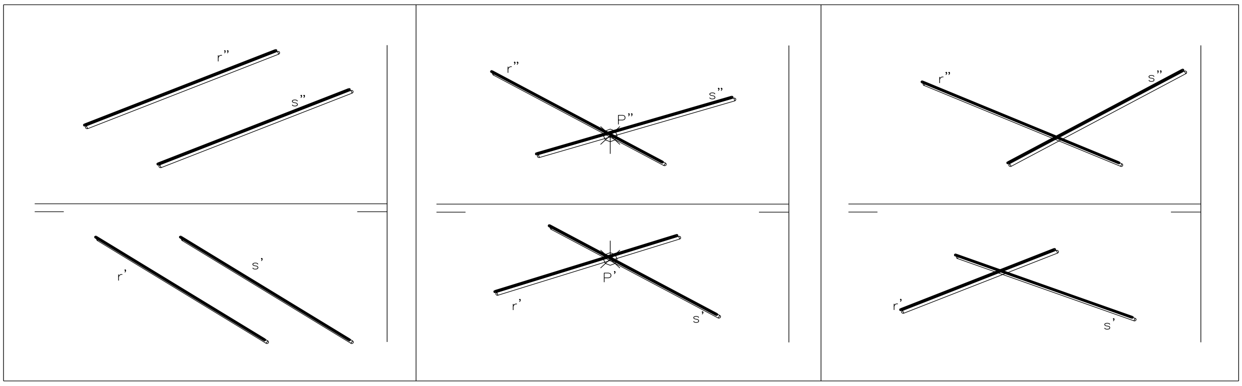
2.1.G – Relative position of two lines

The relative position of two lines can be:

a) They are parallel (their homonymous projections are parallel).

b) They intersect (they have a common point).

c) They skew in the space.

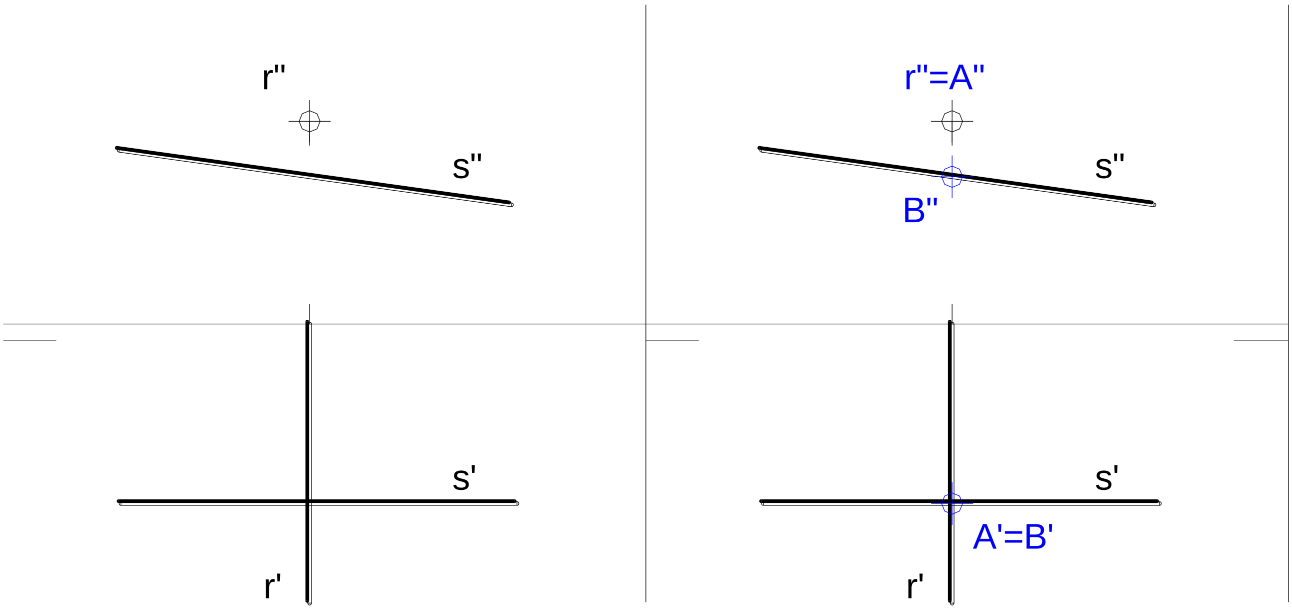


a) b) c)

►*Example 4 (G)*

Given two lines, r and s, define their relative position and discuss the answer.

Solution: Two lines skew in the space because their vertical projections do not have any point in common.



2.1.A – Relative position of two lines

According to basic geometry, it is known that the possible cases for the relative positions of two lines in the space are:

a) The lines intersect: they have exactly one point of intersection.
b) The lines are parallel: they have no points in common and the lines are coplanar.
c) The lines skew: they do not intersect and there is no plane that contains them.
d) The lines coincide.

Next we will analyse the necessary and sufficient conditions of each the above cases by doing an analytic analysis:

2.1.A.1.- Relative position of two lines determined by their implicit equations

Let two lines, r and s, be given in their implicit form:

$$r: \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases} \qquad s: \begin{cases} A_3x + B_3y + C_3z + D_3 = 0 \\ A_4x + B_4y + C_4z + D_4 = 0 \end{cases}$$

In order to determine the relative position of these lines, it is necessary to consider the equations of both lines as a system of linear equations:

$$r \cap s: \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \\ A_3x + B_3y + C_3z + D_3 = 0 \\ A_4x + B_4y + C_4z + D_4 = 0 \end{cases}$$

being M the coefficient matrix and M' the augmented matrix of the given system:

$$M = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \\ A_4 & B_4 & C_4 \end{pmatrix} \qquad M' = \begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{pmatrix}$$

As the first two planes and the last two planes intersect, the rank of matrix M is at least 2. There are different cases depending on the ranks of M and M', which are reduced to the discussion of the system:

	rank M	*rank M'*	*System*	*Relative position of the lines*
Case 1	3	4	Incompatible	Skew lines
Case 2	3	3	Determinate compatible	Lines that intersect
Case 3	2	3	Incompatible	Parallel lines
Case 4	2	2	Indeterminate compatible	The lines coincide

2.1.A.2.- Relative position of two lines determined by their parametric equations

Let two lines, r and s, be given by their parametric equations:

$$r:\begin{cases} x = x_1 + \lambda a_1 \\ y = y_1 + \lambda b_1 \\ z = z_1 + \lambda c_1 \end{cases} \qquad s:\begin{cases} x = x_2 + \mu a_2 \\ y = y_2 + \mu b_2 \\ z = z_2 + \mu c_2 \end{cases}$$

being $A_r = (x_1, y_{1,}, z_1)$ and $A_s = (x_2, y_{2,}, z_2)$ two points of the lines r and s respectively. On the other hand, $\vec{v}_r = (a_1, b_{1,}, c_1)$ is the direction vector of r and $\vec{v}_s = (a_2, b_{2,}, c_2)$ the direction vector of s. We build the vector $\overrightarrow{A_rA_s} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

Depending on the linear dependency of the vectors $\vec{v}_r$, $\vec{v}_s$ and $\overrightarrow{A_rA_s}$, the following cases can be distinguished:

Case 1: $rank(\vec{v}_r, \vec{v}_s) = 2 \neq rank\left(\vec{v}_r, \vec{v}_s, \overrightarrow{A_rA_s}\right) = 3$

The vectors $\vec{v}_r$ and $\vec{v}_s$ are not parallel, therefore, the lines intersect or they are skew. As $rank\left(\vec{v}_r, \vec{v}_s, \overrightarrow{A_rA_s}\right) = 3$, the lines are not coplanar, so they are skew.

Case 2: $rank(\vec{v}_r, \vec{v}_s) = 2 = rank\left(\vec{v}_r, \vec{v}_s, \overrightarrow{A_rA_s}\right)$

The vectors $\vec{v}_r$ and $\vec{v}_s$ are not parallel, therefore, the lines intersect or they are skew. As $rank\left(\vec{v}_r, \vec{v}_s, \overrightarrow{A_rA_s}\right) = 2$, the lines are coplanar, so they are intersecting lines with exactly one point of intersection.

The point of intersection can be calculated by solving the linear system obtained after making equal the values of x, y and z of both lines:

$$r \cap s:\begin{cases} x_1 + \lambda a_1 = x_2 + \mu a_2 \\ y_1 + \lambda b_1 = y_2 + \mu b_2 \\ z_1 + \lambda c_1 = z_2 + \mu c_2 \end{cases}$$

After calculating the values of λ and μ, we substitute one of these values in the line that corresponds to the chosen value and the coordinates of the point of intersection are calculated.

Case 3: $rank(\vec{v}_r, \vec{v}_s) = 1 \neq rank\left(\vec{v}_r, \vec{v}_s, \overrightarrow{A_rA_s}\right) = 2$

The vectors $\vec{v}_r$ and $\vec{v}_s$ are parallel, therefore, the lines are parallel too. As $rank\left(\vec{v}_r, \vec{v}_s, \overrightarrow{A_rA_s}\right) = 2$, the lines do not coincide, that is to say they are parallel and different.

Case 4: $rank(\vec{v}_r, \vec{v}_s) = 1 = rank\left(\vec{v}_r, \vec{v}_s, \overrightarrow{A_rA_s}\right)$

The vectors $\vec{v}_r$ and $\vec{v}_s$ are parallel, therefore, the lines are parallel too. As $rank\left(\vec{v}_r, \vec{v}_s, \overrightarrow{A_rA_s}\right) = 1$, the lines coincide.

►***Example 5 (A)***

Find the relative position of these two lines:

$$r:\begin{cases} x = t \\ y = 2 - t \\ z = 1 + 3t \end{cases} \quad \text{and} \quad p:\begin{cases} x = 2 + 2s \\ y = -2s \\ z = 7 + 6s \end{cases}$$

Solution: The points of the given lines and their direction vectors are the following:

$$A_r = (0,2,1), \qquad \vec{v}_r = (1,-1,3), \qquad A_p = (2,0,7), \qquad \vec{v}_p = (2,-2,6)$$

We calculate the vector $\overrightarrow{A_rA_p} = (2,-2,6)$.

We calculate the ranks of the following matrices:

$$rank(\vec{v}_r, \vec{v}_p) = rank\begin{pmatrix} 1 & -1 & 3 \\ 2 & -2 & 6 \end{pmatrix} = 1 \qquad rank(\vec{v}_r, \vec{v}_p, \overrightarrow{A_rA_p}) = rank\begin{pmatrix} 1 & -1 & 3 \\ 2 & -2 & 6 \\ 2 & -2 & 6 \end{pmatrix} = 1$$

As $rank(\vec{v}_r, \vec{v}_p) = 1 = rank(\vec{v}_r, \vec{v}_s, \overrightarrow{A_rA_p})$, both lines coincide.

►***Example 6 (A)***

Using the implicit equations of the lines r and s, calculate the value of the real parameter a so that the lines are skew:

$$\text{r:}\begin{cases} x + y + z = 1 \\ y = 2 \end{cases} \qquad s:\begin{cases} 2ax + y + z = 1/2 \\ y + 2z = 3 \end{cases}$$

Solution:

We calculate the ranks of the coefficient and the augmented matrices of the system $r \cap s$:

$$rank\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2a & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = 3 \text{ , because } \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 2 \neq 0 \Rightarrow h(M) = 3 \ \forall a \in \mathbb{R}$$

We calculate the determinant of the augmented matrix:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 2a & 1 & 1 & 1/2 \\ 0 & 1 & 2 & 3 \end{vmatrix} = 4 - 6a$$

The determinant of the augmented matrix has not to be zero, if we want the lines r and s to be skew. That is to say, the rank of M' has to be 4.

Therefore, $4 - 6a \neq 0 \Rightarrow a \neq 2/3$.

2.1. Mutual examples of both subjects

►*Example 7 (A)*

Given the following lines:

$r: \frac{x-1}{1} = \frac{y-2}{1} = \frac{z-1}{2}$ and $s: \frac{x-3}{-2} = \frac{y-3}{-1} = \frac{z+1}{2}$,

find their relative position and analyse if they have any point in common.

Solution: From these equations, a point and the direction vector of each of the lines are derived:

$$A_r = (1,2,1), \qquad \vec{v}_r = (1,1,2) \qquad A_s = (3,3,-1), \qquad \vec{v}_s = (-2,-1,2)$$

We build the vector $\overrightarrow{A_r A_s} = (2,1,-2)$.

We calculate the ranks:

$$rank(\vec{v}_r, \vec{v}_s) = rank\begin{pmatrix} 1 & 1 & 2 \\ -2 & -1 & 2 \end{pmatrix} = 2$$

$$rank\left(\vec{v}_r, \vec{v}_s, \overrightarrow{A_r A_s}\right) = rank\begin{pmatrix} 1 & 1 & 2 \\ -2 & -1 & 2 \\ 2 & 1 & -2 \end{pmatrix} = 3 \text{ , because } \begin{vmatrix} 1 & 1 & 2 \\ -2 & -1 & 2 \\ 2 & 1 & -2 \end{vmatrix} \neq 0.$$

As $rank\left(\vec{v}_r, \vec{v}_p\right) = 2 \neq rank\left(\vec{v}_r, \vec{v}_s, \overrightarrow{A_r A_p}\right) = 3$ is satisfied, r and s are intersecting lines.

Next we calculate the point of intersection. The parametric equations of each of the lines are given by:

$$r: \begin{cases} x = 1 + t \\ y = 2 + t \\ z = 1 + 2t \end{cases} \text{ and } \quad s: \begin{cases} x = 3 - 2s \\ y = 3 - s \\ z = -1 + 2s \end{cases}$$

And making equal the equations, we get the following linear system:

$$\begin{cases} 1 + t = 3 - 2s \\ 2 + t = 3 - s \\ 1 + 2t = -1 + 2s \end{cases}$$

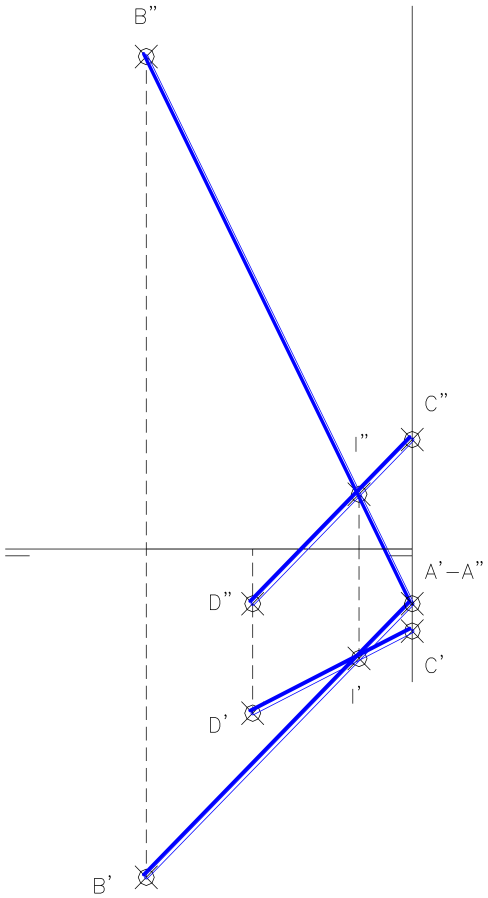
The solution of the system is: $t = 0$ and $s = 1$.

After substituting one of these parameters in the line in which the chosen parameter appears, we obtain the point of intersection $P = (1,2,1)$.

►*Example 7 (G)*

Define the relative position of r and s.

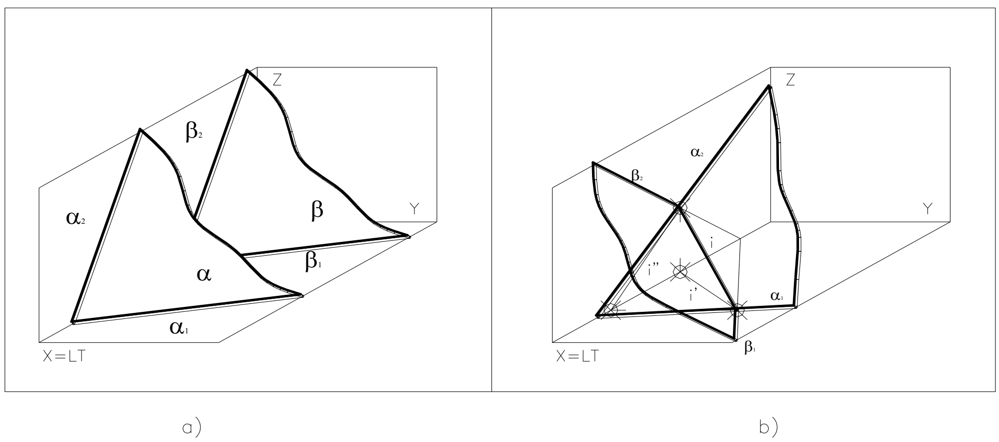
Solution: As the horizontal and vertical projections of the lines coincide at the same point, the two lines intersect.



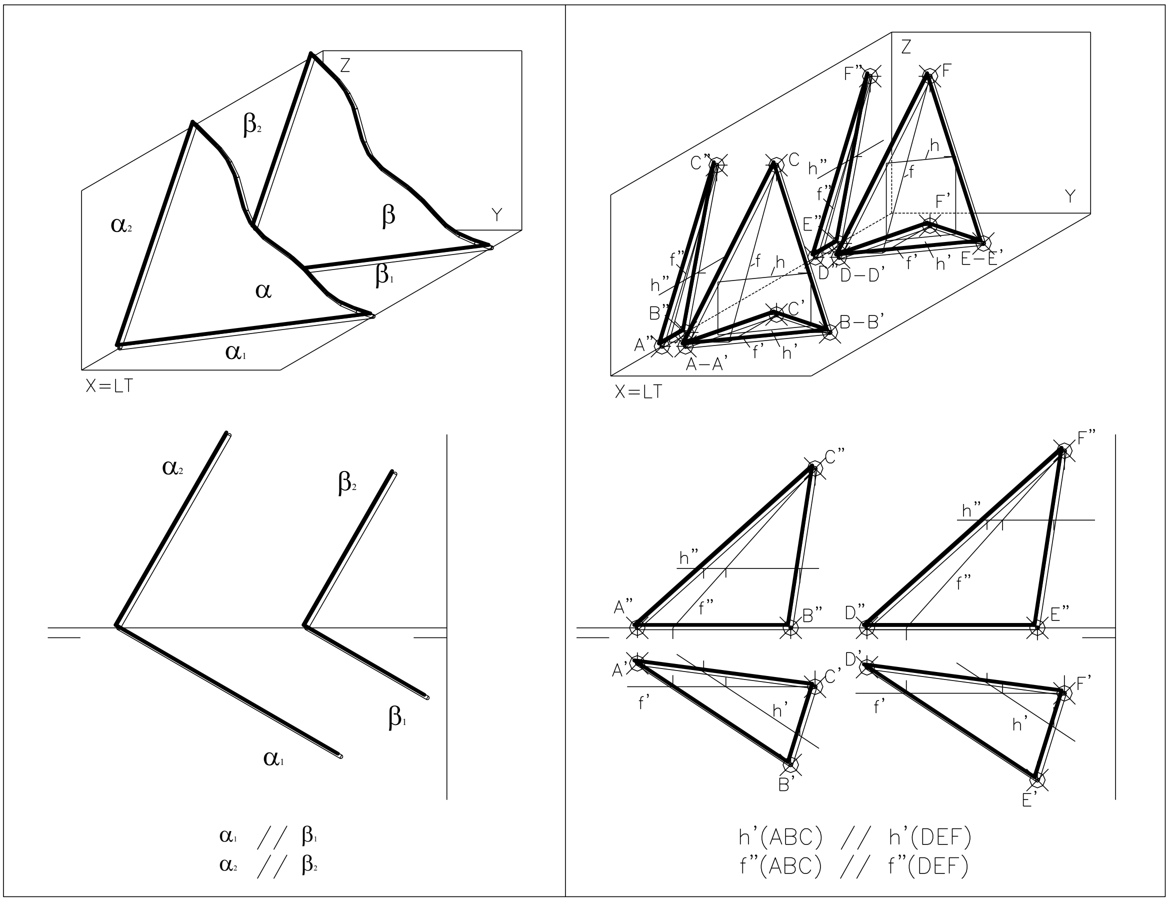
2.2.G – Relative position of two planes

The relative position of two planes could be: they can be parallel or they can intersect.

a) Parallel planes (they do not have any point in common)



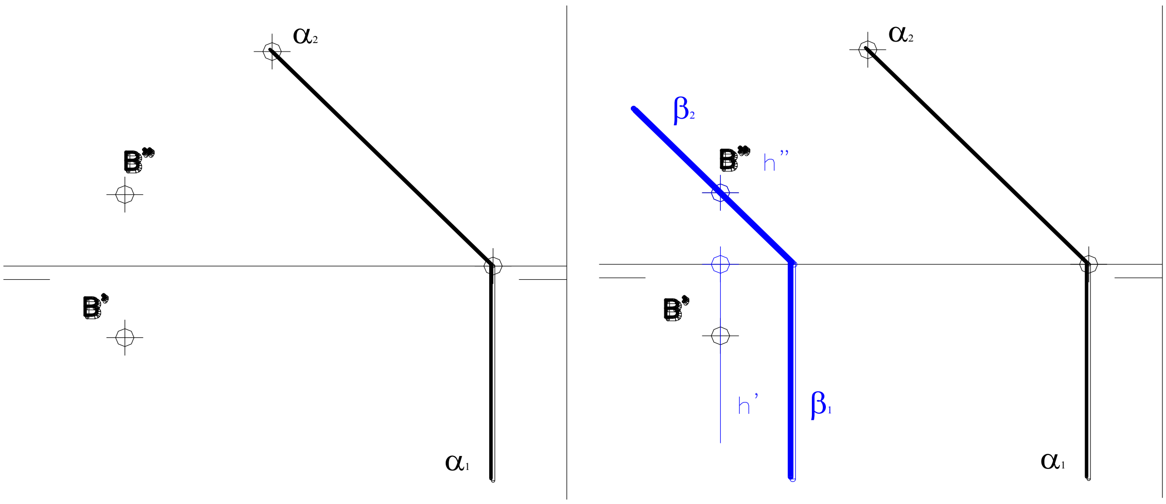
If two planes are parallel, their homonymous traces will be parallel too. In the same way, the homonymous projections of the frontal and horizontal lines of the plane will be parallel.



a)

►*Example 8 (G)*

Given the point B, define a plane that passing through this point is parallel to α.

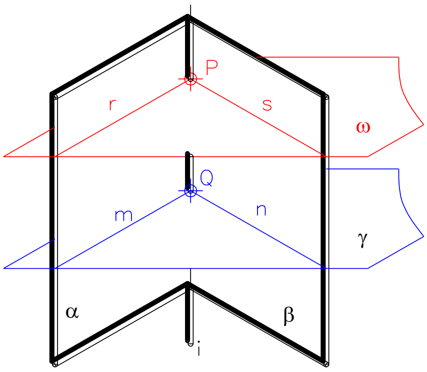


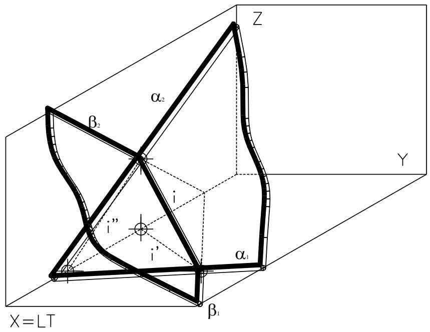
b) The planes intersect in a straight line (the line “i” in the image).

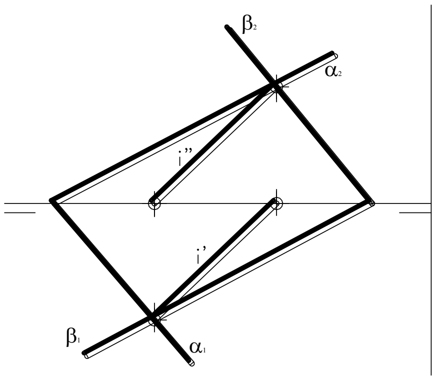
In general, to find the intersection-line of two planes, two auxiliary planes are necessary.

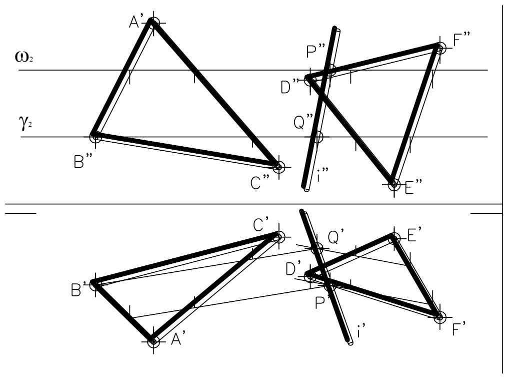
- If the traces of the plane are known, we will select the projection planes XY and XZ as auxiliary planes. Thus, it will be enough to find the intersection of the traces.

- If the traces are not known, we will use planes that are parallel to the projection planes.

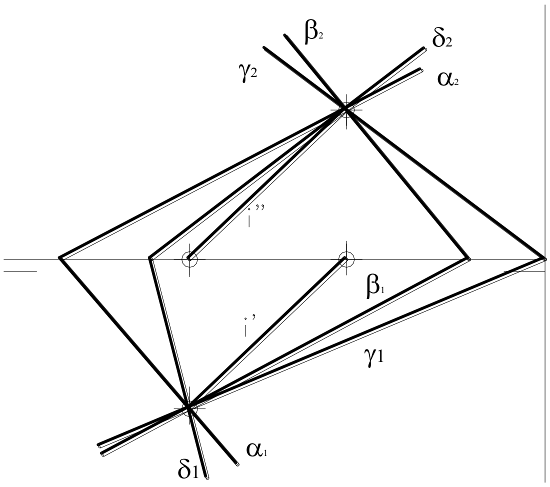


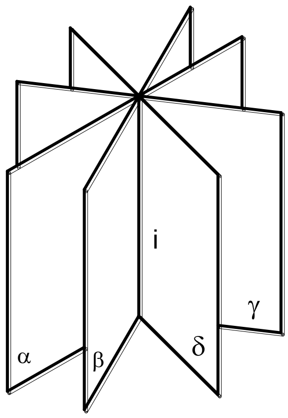






The set of planes that intersect in the same line, is called sheaf of intersecting planes.





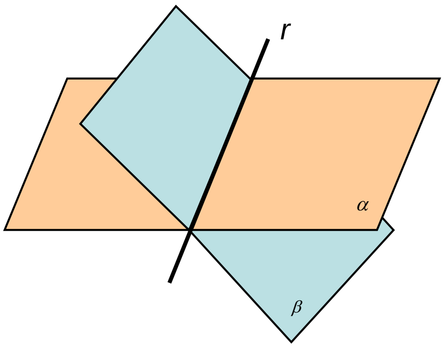
2.2.A – Relative position of two planes

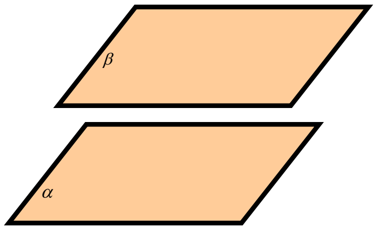
According to basic geometry, it is known that the possible cases for the relative positions of two planes are:

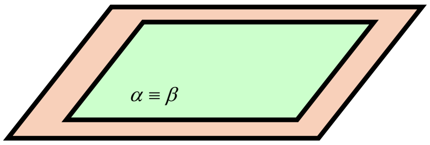
a) The planes intersect in a line.

b) The planes are parallel: they have no points in common.

c) The planes coincide.







Next we will analyse the necessary and sufficient conditions of each the above cases by doing an analytic analysis depending on the form in which the plane is defined.

Analytic expression

Let two planes be given by their general equations:

$$\begin{aligned} \alpha: A_1x+B_1y+C_1z+D_1 &= 0 \\ \beta: A_2x+B_2y+C_2z+D_2 &= 0 \end{aligned} \qquad (1)$$

The relative position of these two planes is studied considering the linear system formed by their general equations. The coefficient matrix and augmented matrix of the linear system are the following:

$$M=\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} \quad M'=\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \end{pmatrix}$$

Depending on the ranks of the matrices M and M', the following cases can be distinguished:

<u>Case 1:</u> rank(M)=2 and rank(M')=2

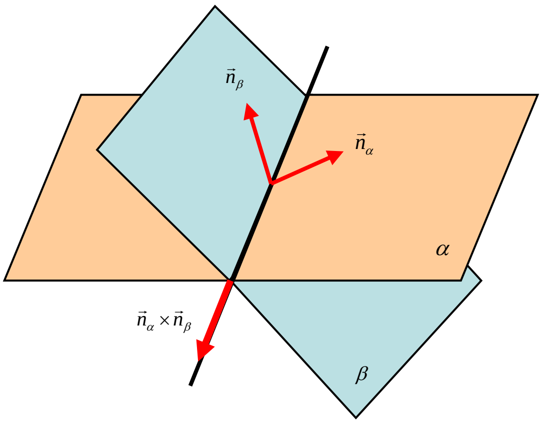
The system is compatible indeterminate with one grade of indetermination. That is to say, the infinite solutions of this system depend on one parameter. Therefore, they are intersecting planes being the locus of these intersecting planes exactly one line.

This means that a line can also be defined as the intersection of two planes.

The equations that form the system of two intersecting planes are the implicit equations of a line.

The direction vector of the line of intersection r is given by:

$$\vec{u}_r=\vec{n}_\alpha\times\vec{n}_\beta=(A_1,B_1,C_1)\times(A_2,B_2,C_2)\,.$$



A point of the intersection line can be calculated by solving the system of two unknowns obtained after giving a value to one of the unknowns x, y or z.

As a point of the line and its direction vector are known, the line can be written in one of its forms (vector equation, parametric, or continuous equation).

<u>Case 2:</u> rank(M)=1 and rank(M')=2

Incompatible system. The planes are different (no points in common) and parallel.

<u>Case 3:</u> rank(M)=1 and rank(M')=1

Compatible indeterminate system. The equations of the planes are proportional to each other. Then the plane α is just the same as β.

	rank M	*rank M'*	*System*	*Relative position of the planes*
Case 1	2	2	Compatible indeterminate	Planes that intersect
Case 2	1	2	Incompatible	Parallel planes
Case 3	1	1	Compatible indeterminate	The planes coincide

In practice, the relative position between two planes is analysed by studying the proportionality of the coefficients and the non-homogeneous terms of the equations of the planes:

1. If the coefficients are not proportional, the planes intersect.
2. If the coefficients are proportional but the non-homogeneous terms not, the planes are parallel.
3. If both the coefficients and the non-homogeneous terms are proportional, the planes coincide.

Sheaf of parallel planes

Given the implicit equation of a plane:

$$Ax + By + Cz + D = 0,$$

the parallel planes to the given plane have this equation:

$$Ax + By + Cz + K = 0, K \in \mathbb{R}$$

Parallel planes have the same normal vector, in this case $\vec{n} = (A, B, C)$.

The set of planes that are parallel to a given plane is called sheaf of parallel planes. This sheaf of parallel planes is determined by a plane that belongs to the sheaf and the equation of this plane is given by:

$$Ax + By + Cz + K = 0, K \in \mathbb{R}$$

A sheaf of planes is used in the following cases to determine the equation of a plane:

a) To calculate the plane that passing through the point P(x_1, y_1, z_1) is parallel to the plane $Ax + By + Cz + D = 0$.

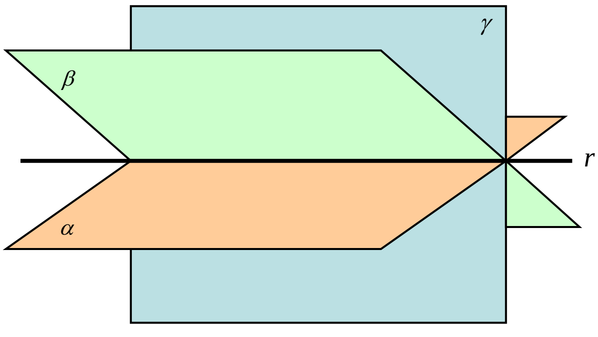
b) To calculate the plane that passing through the point P(x_1, y_1, z_1) is perpendicular to the vector $\vec{n} = (A, B, C)$.

c) To calculate the plane that passing through the point P(x_1, y_1, z_1) is perpendicular to the line $\dfrac{x-A}{A} = \dfrac{y-b}{B} = \dfrac{z-c}{C}$.

Sheaf of intersecting planes

If two given planes intersect in the line r, and a third plane passes through this line, the common solutions of the first two planes will be also included in the third plane. This means that the third plane is a linear combination of the first two planes and it can be written as:

$$A_3x + B_3y + C_3z + D_3 = t(A_1x + B_1y + C_1z + D_1) + s(A_2x + B_2y + C_2z + D_2) = 0$$



By doing $s = 0$, the first plane is obtained and when $t = 0$, the second. In the same way, any plane passing through the intersection line of the first two planes has the same solutions.

The set of all planes that have the same common line is called sheaf of intersecting planes. And this common line r is called the edge of the sheaf of planes.

A sheaf of planes is determined by two planes of the sheaf, and it is given by:

$$t(A_1x + B_1y + C_1z + D_1) + s(A_2x + B_2y + C_2z + D_2) = 0, \qquad t, s \in \mathbb{R}$$

The previous expression can be written as follows symbolically $t\alpha + s\beta = 0$, being α and β two planes of the sheaf.

Determination of planes by means of sheafs

A typical problem of a sheaf of intersecting planes is the problem in which we want to calculate the plane that passing through the point $P(x_1, y_1, z_1)$ contains the intersection line of the planes α and β :

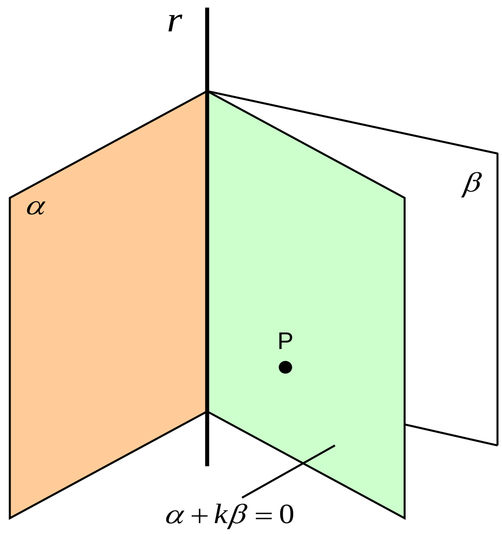
$$\begin{aligned} \alpha &: A_1x + B_1y + C_1z + D_1 = 0 \\ \beta &: A_2x + B_2y + C_2z + D_2 = 0 \end{aligned}$$

We assume that the point does not belong to the planes α and β. If the plane that we want to calculate contains the intersection line of these two planes, it will be a plane of the sheaf of planes, and it will be given by the next expression: $t\alpha + s\beta = 0$.

By substituting the coordinates of the point P in the equation of the sheaf, the relationship between the parameters t and s can be calculated:

$$t(A_1x_1 + B_1y_1 + C_1z_1 + D_1) + s(A_2x_1 + B_2y_1 + C_2z_1 + D_2) = 0$$

After some calculations, a relationship of the form $s = kt$ is obtained, being k a real number. By substituting this value in the sheaf, the equation of the plane that we are seeking is obtained: $t\alpha + kt\beta = 0 \Rightarrow \alpha + k\beta = 0$



►*Example 9 (A)*

Find the relative position of these two planes:

$$\begin{aligned} \alpha &: x+y-5z &&= -4 \\ \beta &: -3x-3y+15z &&= 12 \end{aligned}.$$

Solution: Considering the linear system formed by both equations, the ranks of the matrices are: rank(M)=1 and rank(M')=1. So, the system is compatible. In addition, both equations are proportional, and the system is reduced to a unique equation. This means that both planes coincide.

2.2. Mutual examples of both subjects

►*Example 10 (A)*

Determine the relative position of these planes:

$$\begin{aligned} \alpha &: x+y-5z &&= -4 \\ \beta &: 3x-y+25z &&= 1 \end{aligned}$$

Solution: As the ranks of the matrices satisfy rank(M)=rannk(M')=2, the linear system formed by the equations of the planes is compatible, and the planes intersect in one line. The direction vector of this line can be calculated doing the vector product of the normal vectors of the planes α and β : $\vec{n}_\alpha \times \vec{n}_\beta = 3\vec{i} - 17\vec{j} - 4\vec{k}$.

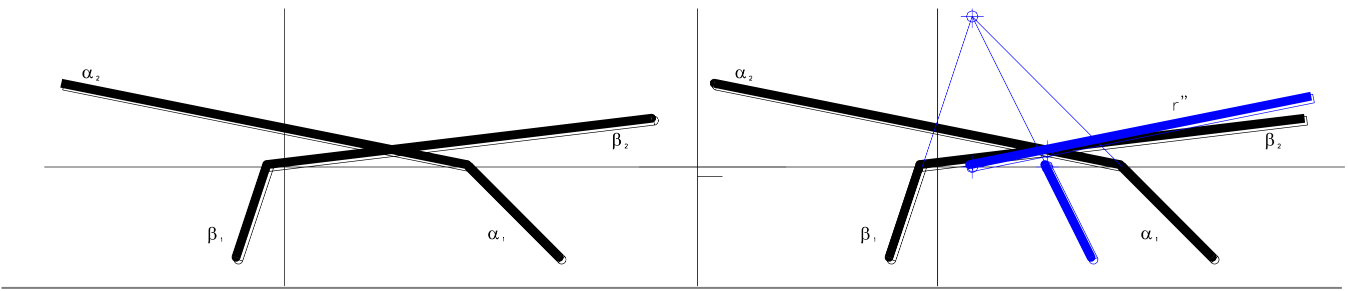
Finally, it is enough to consider one point of the intersection to determine the line of intersection of both planes. We solve the linear system obtained after fixing $z=5$ in both planes, and we obtain $x=3$ and $y=18$. Therefore, the line is given in its continuous form as follows:

$$r: \frac{x-3}{3} = \frac{y-18}{-17} = \frac{z-5}{-4}$$

►*Example 10 (G)*

Define the relative position of the planes α and β.

Solution: Two planes intersect in the line r.



►***Example 11 (A)***

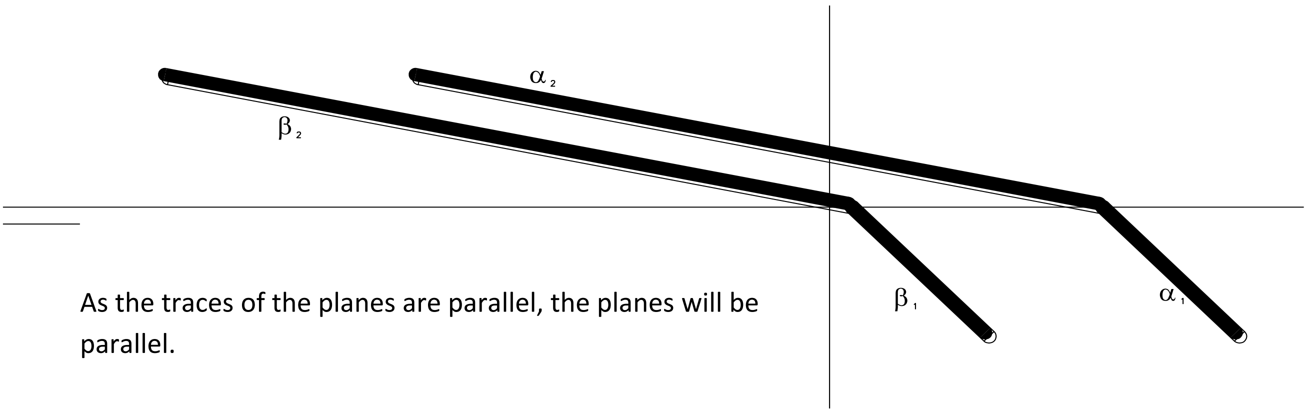
Find the relative position of these two planes:

$$\begin{cases} \alpha: x+y-5z & =-4 \\ \beta: -3x-3y+15z & =1 \end{cases}.$$

Solution: As the ranks of the matrices are different rank(M)=1 and rank(M')=2, the linear system formed by the equations of the planes is incompatible. But the non-homogeneous terms of the equations are not proportional, so the planes are parallel (and different).

►***Example 11 (G)***

Define the relative position of the planes α and β.



As the traces of the planes are parallel, the planes will be parallel.

►***Example 12 (A)***

Find a plane that passing through the point $A(1,1,1)$ is parallel to the plane $3x-5y+z-5=0$.

*Solution: T*he planes that are parallel to the given plane have the next expression:

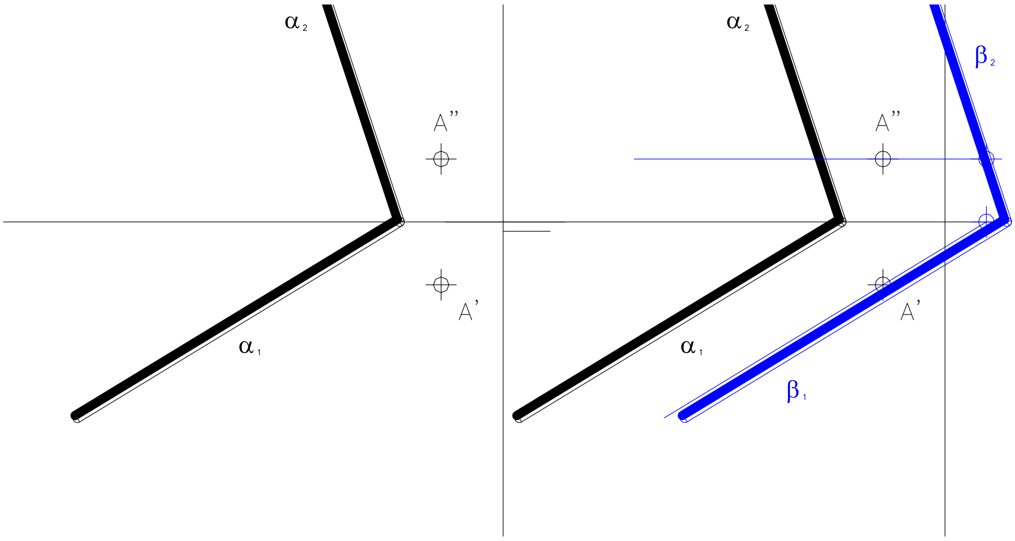
$3x-5y+z+K=0$

On the other hand, as the point $A(1,1,1)$ is included in the plane: $3-5+1+K=0 \Rightarrow K=1$

So the plane that satisfies the required conditions is given by: $3x-5y+z+1=0$.

►*Example 12 (G)*

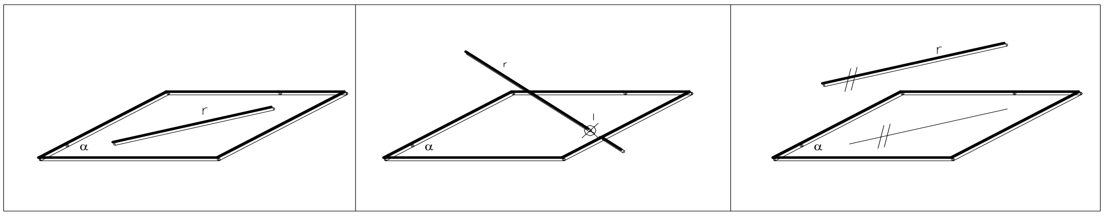
Draw from the point A a plane that is parallel to α.



2.3.G – Relative position of a line and a plane

The relative position of a line and a plane can be:

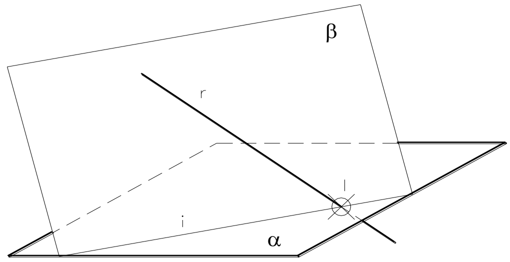
a) The line is inside the plane (all the points of the line are located in the plane).
b) The line and the plane intersect in a point.
c) The line and the plane are parallel (they have not any point in common).



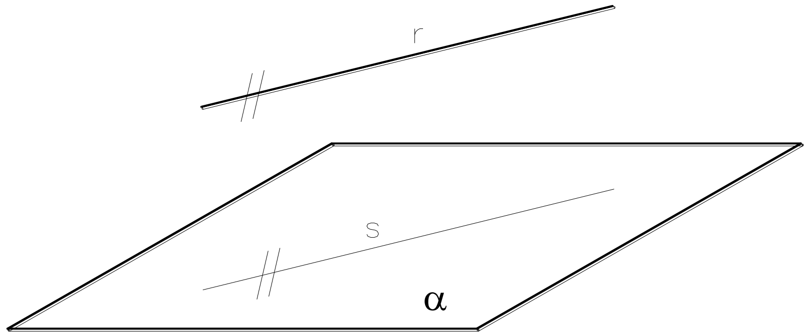
a) b) c)

a) As we have seen before, a line is located in a plane if its traces are in the homonymous traces of the plane.

b) Intersection between a line and a plane: to define the intersection between a line r and a plane α we need an auxiliary plane β. This auxiliary plane is usually a projecting plane that contains the line r. The intersection between the intersection-line (i) between the planes α and β and the line r (that is located in the same plane) will be the intersection point (I).

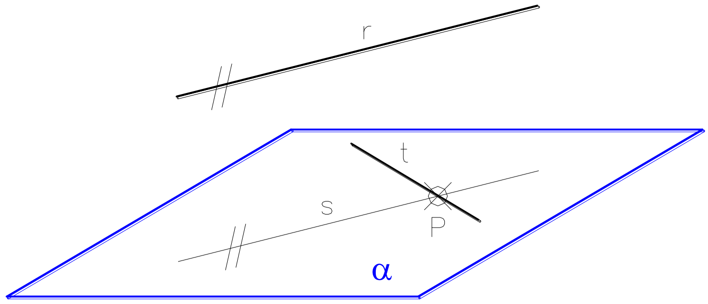


c) The line r will be parallel to the plane α if it is parallel to at least one line located in the plane.



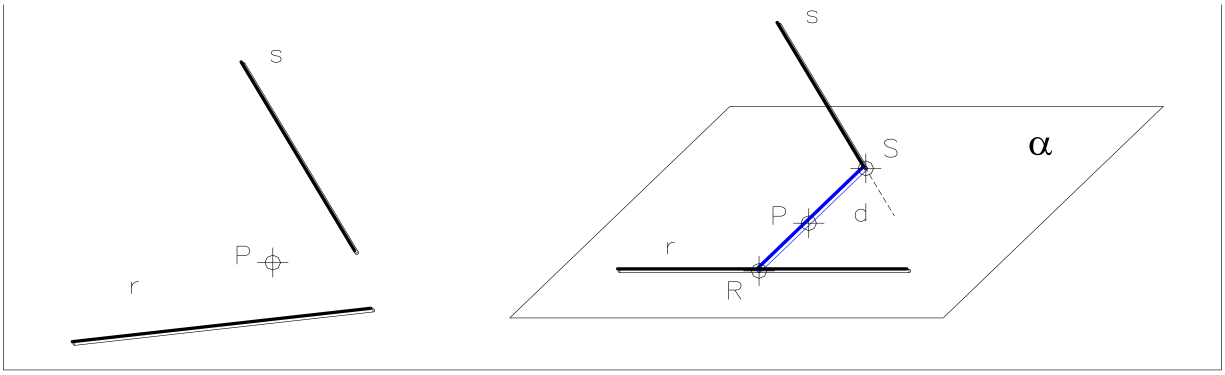
►***Exercise***

Given the lines r and s, define a plane that passing through a line t is parallel to the line r.
The procedure that must be followed is:
1. From any point (P) of the line t, draw a line (s) that is parallel to the line r.
2. Draw the plane that is defined by s and t (that is the solution).



►***Exercise***

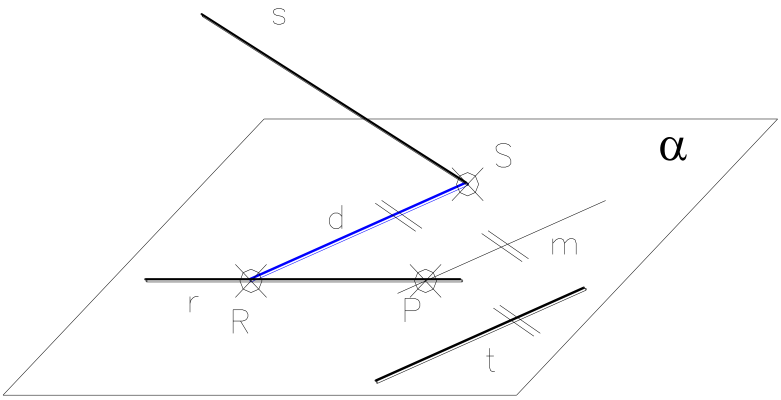
Given the lines r and t, draw another line from P that intersects these two lines.
The procedure that must be followed is:
1. Draw the (α) plane that is defined by the line r and the point P.
2. Find the intersection (S) between the line s and the plane α.
3. The point S and P will define the line we are looking for, because these two points, together with the line r, are in the same plane. As these two lines are in the same plane, they will intersect or they will be parallel. In this case they intersect in the point R.



► ***Exercise***

Given the lines r, s and t, draw a line that is parallel to t and intersects the lines r and s. The procedure that must be followed is:

1. From any point (P) of the line r, draw a line (m) that is parallel to the line t.
2. Draw the plane (α) that is defined by r and m.
3. Find the intersection between the plane α and the line s: the point S.
4. From the point S, draw a line (d) that is parallel to the line t.
5. Find the intersection between the lines d and r: the point R.
6. The line d is the solution we are looking for.



2.3.A – Relative position of a line and a plane

2.3.A.1.- Relative position of a line and a plane defined by their implicit equations

Let a line be determined by $r:\begin{cases} A_1x+B_1y+C_1z+D_1=0 \\ A_2+B_2y+C_2z+D_2=0 \end{cases}$ and a plane $\pi: A_3+B_3y+C_3z+D_3=0$.

To investigate the relative positions of the plane and the line, the following system of equations is considered:

$$r\cap\pi:\begin{cases} A_1x+B_1y+C_1z+D_1=0 \\ A_2+B_2y+C_2z+D_2=0 \\ A_3+B_3y+C_3z+D_3=0 \end{cases}$$

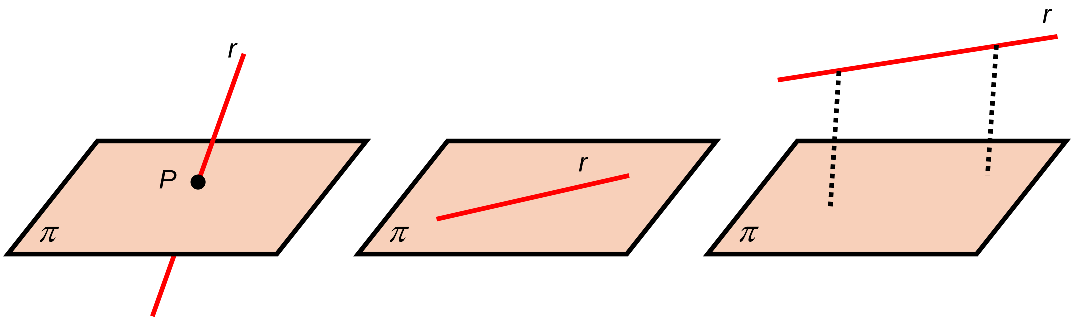
being $M=\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$ the coefficient matrix and $M'=\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{pmatrix}$ the augmented matrix.

There are three possible cases for the system $r \cap \pi$:

Case 1: Compatible determinate ($rank(M)=rank(M')=3=$number of unknowns). The system has a unique solution. It means that the plane and the line intersect in a point, being this point the solution of the linear system. In this case we say that the line r and the plane π intersect.

Case 2: Compatible indeterminate ($rank(M)=rank(M')=2<3=$number of unknowns). The system has infinite solutions because the third equation is linear combination of the first two equations. In this case, the plane π belongs to the sheaf of planes with edge r. Therefore, $r \subset \pi$. In this case, the line r lies in the plane π.

Case 3: Incompatible ($rank(M)=2 \neq rank(M')=3$) The system has no solution. Therefore, $r \cap \pi = \{\varnothing\}$ and the line r is parallel to the plane π.



The line and the plane intersect in a point.

The line lies in the plane.

The line and the plane are parallel.

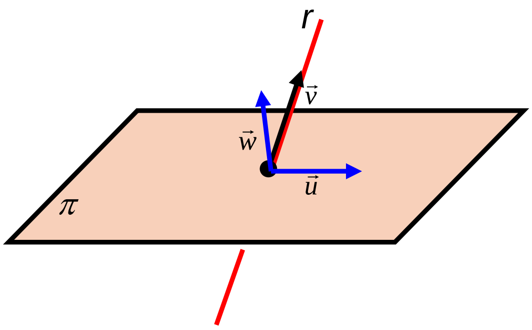
2.3.A.2- Relative position of a line and a plane defined by their vector equations or by their parametric equations

Let a line r be given by its parametric equations: $r:\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}+\lambda\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, being $P=(p_1,p_2,p_3)$ a point of the line and $\vec{v}=(v_1,v_2,v_3)$ the direction vector of the line.

Let a plane π be given by its parametric equations: $\pi:\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}+\lambda\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}+\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$, being $Q=(q_1,q_2,q_3)$ a point in the plane and $\vec{u}=(u_1,u_2,u_3)$ and $\vec{w}=(w_1,w_2,w_3)$ the direction vectors of the plane. The relative position of the line and the plane can be determined depending on their direction vectors:

a. <u>The line and the plane intersect in a point</u>

If the direction vector of the line and the direction vectors of the plane are linearly independent, the line and the plane intersect in a point. In this case, $rank\begin{pmatrix} v_1 & u_1 & w_1 \\ v_2 & u_2 & w_2 \\ v_3 & u_3 & w_3 \end{pmatrix}=3$ is satisfied.

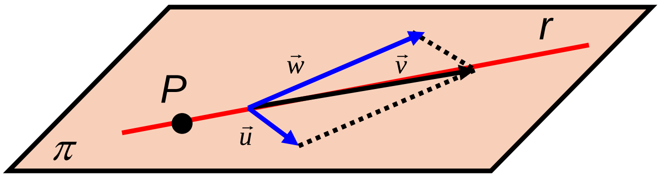


b. <u>The line lies in the plane or they are parallel</u>

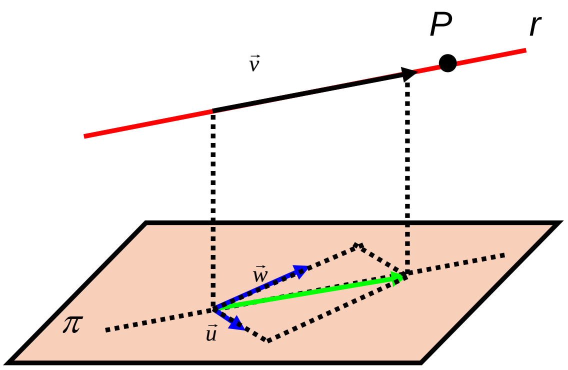
If the direction vector of the line is linear combination of the direction vectors of the plane, there are two possibilities: the line lies in the plane or they are parallel. In both cases $rank\begin{pmatrix} v_1 & u_1 & w_1 \\ v_2 & u_2 & w_2 \\ v_3 & u_3 & w_3 \end{pmatrix}=2$ is satisfied.

Both cases can be distinguished by substituting in the plane any point of the line. If the point satisfies the equation of the plane, the line lies in the plane. Otherwise, the line and the plane are parallel.

The line lies in the plane



The line and the plane are paralllel



► ***Example 13 (A)***

Discuss the relative position between the plane $\alpha: x+ay-z=1$ and the line $r:\begin{cases} 2x+y-az=2 \\ x-y-z=a-1 \end{cases}$ depending on the real parameter a.

Solution: The following system has to be studied: $r\cap\alpha: \begin{cases} x+ay-z-1=0 \\ 2x+y-az-2=0 \\ x-y-z-a+1=0 \end{cases}$

being $M=\begin{pmatrix} 1 & a & -1 \\ 2 & 1 & -a \\ 1 & -1 & -1 \end{pmatrix}$ and $M'=\begin{pmatrix} 1 & a & -1 & -1 \\ 2 & 1 & -a & -2 \\ 1 & -1 & -1 & 1-a \end{pmatrix}$ the coefficient matrix and the augmented matrix respectively.

As $\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}=-3\neq 0 \Rightarrow rank\,(M)\geq 2$.

$$|M|=\begin{vmatrix} 1 & a & -1 \\ 2 & 1 & -a \\ 1 & -1 & -1 \end{vmatrix}=-a^2+a+2 \quad -a^2+a+2=0 \Rightarrow (a-2)(a+1)=0$$

We start the discussion depending on the values of the parameter:

- If $a\neq 2$ and $a\neq -1 \Rightarrow rank(M)=3=rank(M')\Rightarrow$ the line and the plane intersect exactly in a point.

- If $a=2$:

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 1 & -1 & -1 \end{vmatrix}=0 \Rightarrow rank(M)=2=rank\left(M'\right)\Rightarrow$$ The line lies in the plane.

- If $a=-1$:

$$\begin{vmatrix} 1 & -1 & -1 \\ 2 & 1 & -2 \\ 1 & -1 & 2 \end{vmatrix} = 9 \neq 0 \Rightarrow rank(M) = 2 \neq rank\left(M'\right) = 3 \Rightarrow$$ The line is parallel to the plane.

2.3 Mutual examples of both subjects

►***Example 14 (A)***

Determine the relative position between the line $r: \dfrac{x}{1} = \dfrac{y}{1} = \dfrac{z+1}{1}$ and the plane $\alpha: 2x-3y+z+1=0$.

Solution: We will start by calculating the implicit equations of the line:

$$r: \begin{cases} x = y \\ x = z+1 \end{cases} \Rightarrow r: \begin{cases} x-y=0 \\ x-z-1=0 \end{cases}$$

The following system is defined: $r \cap \alpha: \begin{cases} 2x-3y+z+1=0 \\ x-y=0 \\ x-z-1=0 \end{cases}$

being $M = \begin{pmatrix} 2 & -3 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$ and $M' = \begin{pmatrix} 2 & -3 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & -1 \end{pmatrix}$ the coefficient matrix and the augmented matrix respectively.

We will study the ranks of M and M':

$$\begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix} = 1 \neq 0 \Rightarrow rank\left(M\right) \geq 2$$

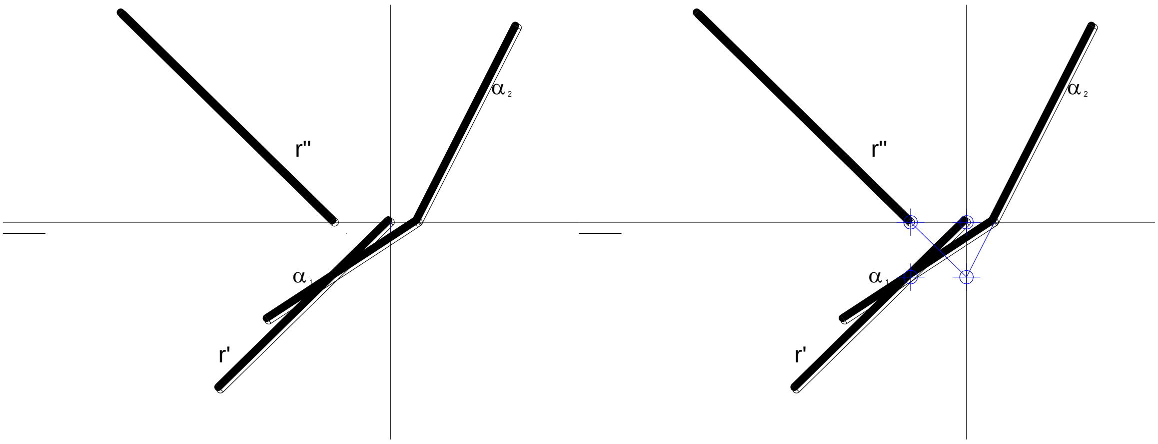
$$\left|M\right| = \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 0 \Rightarrow rank\left(M\right) = 2 = rank\left(M'\right)$$

Therefore, the line lies in the plane.

►Example 14 (G)

Verify the position between the line r and the plane α.

Solution: As we can see in the image, the traces of the line are located in the homonymous traces of the plane. Thus, the line is in the plane.



►Example 15 (A)

Calculate the value of the real parameter b so that the line $r: \frac{x}{1} = \frac{y-2}{b} = \frac{z+3}{2}$ does not intersect the plane $\alpha: 2x - 4y + 5z = 6$.

Solution:

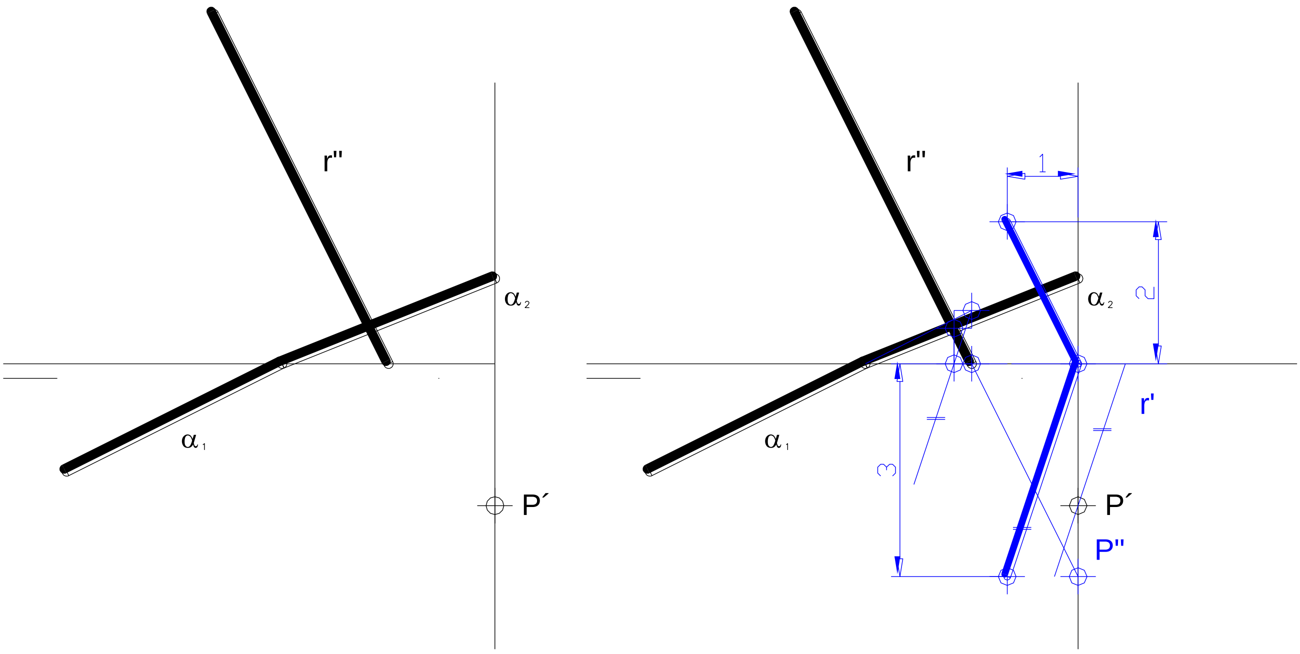
The line will not intersect the plane if they are parallel. This means that the direction vector of the line and the normal vector of the plane will be perpendicular (their scalar product will be zero).

The direction vector of the line is $\vec{u} = (1, b, 2)$ and $\vec{a} = (2, -4, 5)$ is the normal vector of the plane. We can calculate the value of the parameter b by making equal to zero the scalar product of both vectors:

$$\vec{u} \perp \vec{a} \Rightarrow \vec{u} \cdot \vec{a} = 0 \Rightarrow (1, b, 2) \cdot (2, -4, 5) = 0 \Rightarrow 2 - 4b + 10 = 0 \Rightarrow b = 3$$

►*Example 15 (G)*

Find r' and P'', so that r will be parallel to α. Define the parameter that is missing.



►*Example 16 (A)*

Find the intersection between the line $r: \begin{cases} x+2y=9 \\ x+2z=7 \end{cases}$ and the plane $\alpha: x-13y-8z+41=0$.

Solution: The intersection of the line and the plane can be calculated by solving the following linear system:

$$r\cap\alpha: \begin{cases} x+2y=9 \\ x+2z=7 \\ x-13y-8z=-41 \end{cases} \Rightarrow x=\frac{91}{23}\ ,\ y=\frac{58}{23}\ ,\ z=\frac{35}{23}$$

►*Example 16 (G)*

Define the intersection between the line r and the plane.