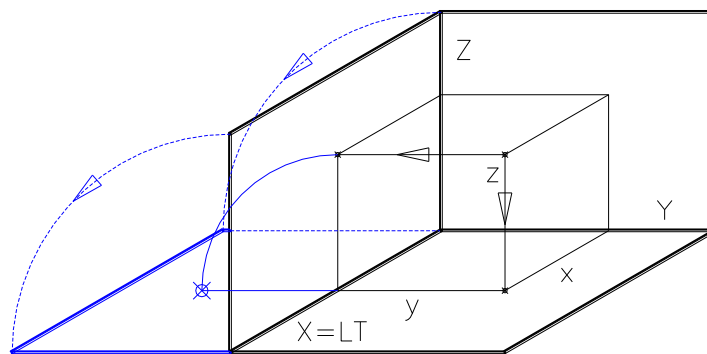


# LESSON I: ELEMENTS OF THE AFFINE SPACE: DEFINITION AND REPRESENTATION

## 1.1.G - Reference system

In the Diedric System we use three orthogonal planes (XY, XZ and ZY), called PH, PV and PP, to project and represent the position of each point. In general, it is enough to use PH and PV, and we use PP only in special cases.

The intersection between PH and PV is called floor-line (LT). It is possible to work only in a plane, by folding one plane around the floor-line until the other plane is obtained.

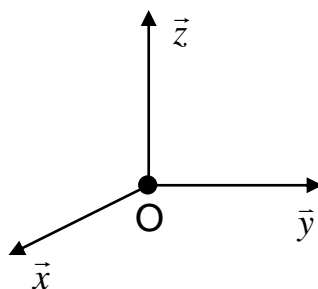


## 1.1.A - Reference system

Let be  $O$  a fixed point in the space and  $B = \{\vec{x}, \vec{y}, \vec{z}\}$  a basis of  $\mathbb{R}^3$ . The position of any point of the space is defined by means of the point  $O$  and the basis  $B = \{\vec{x}, \vec{y}, \vec{z}\}$ . Therefore, the point  $O$  and the basis  $B = \{\vec{x}, \vec{y}, \vec{z}\}$  form a reference system of the space, denoted by  $R = \{O ; \vec{x}, \vec{y}, \vec{z}\}$ .

The point  $O$  is the origin, and the lines that passing through the origin are parallel to the vectors of the basis are the coordinate axes, which are denoted by  $OX, OY$  and  $OZ$  respectively.

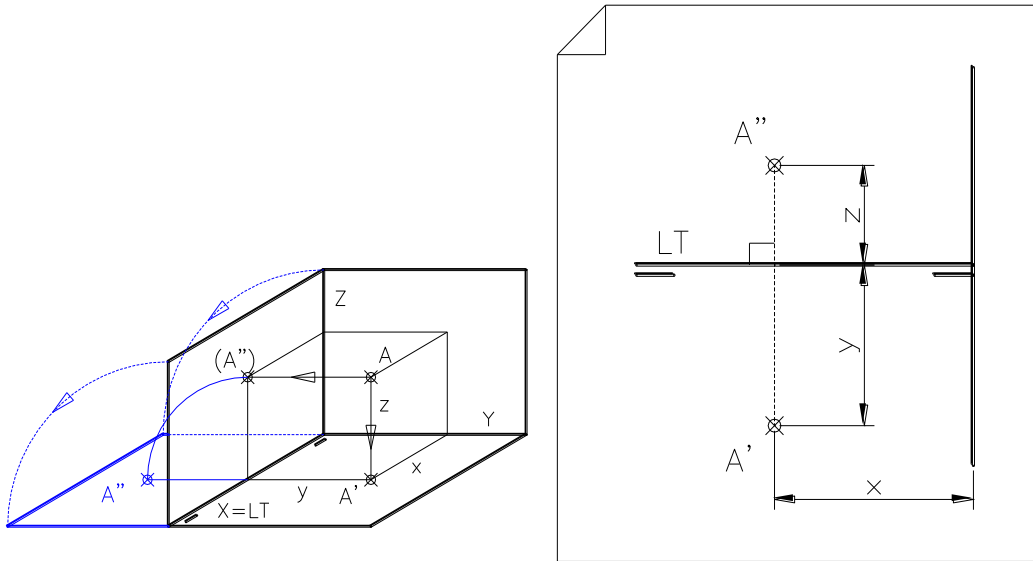
We will use the orthonormal basis, which means that the vectors  $\vec{x}, \vec{y}$  and  $\vec{z}$  will be unit vectors (their modulus is 1) and perpendicular to each other.



### 1.2.G - Point

In the diedric system one point (A) is defined by its two projections (A' in the PH and A'' in the PV). The line that joins two projections is, by definition, perpendicular to the floor-line (LT). The Cartesian coordinates (x, y, z) of the point will be:

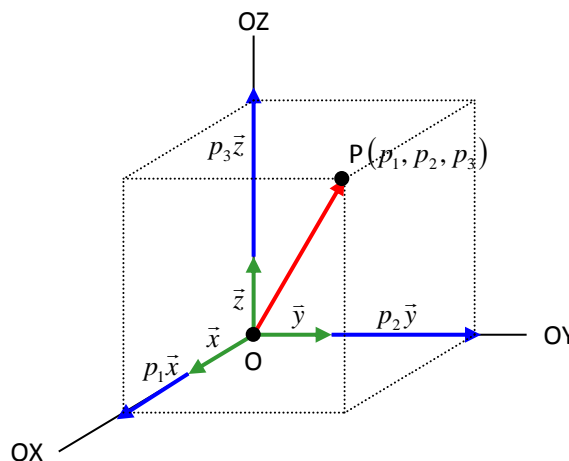
- $x$  = In the work-space (XY+XZ planes) the distance to the YZ reference plane (called profile plane) that is perpendicular to the floor-line (LT).
- $y$  = the distance from the A' projection to the LT.
- $z$  = the distance from the A'' projection to the LT.



### 1.2.A – Coordinates of a point

Any point  $P$  of the space, together with the origin  $O$  determines the vector  $\overrightarrow{OP}$  of coordinates  $(p_1, p_2, p_3)$  with respect to the basis  $B$ . That is to say,  $\overrightarrow{OP} = p_1\vec{x} + p_2\vec{y} + p_3\vec{z}$ . Therefore, the cartesian coordinates of the point  $P$  with respect to the reference system

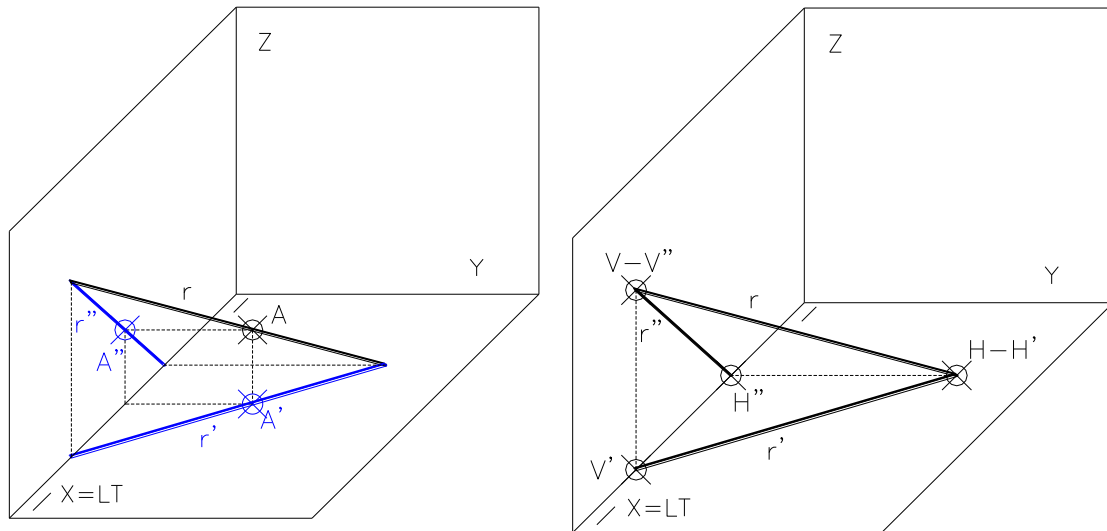
$R = \{O ; \vec{x}, \vec{y}, \vec{z}\}$  are given by  $(p_1, p_2, p_3)$ .



### 1.3.G- Line

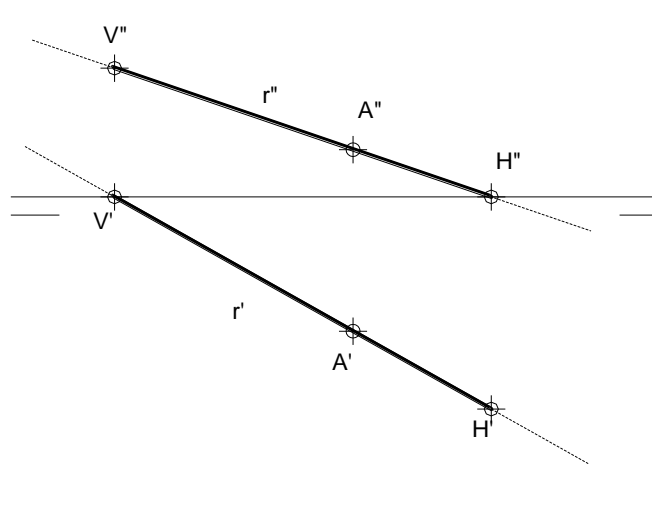
The two projections of a line ( $r$ ) ( $r'$  in the PH plane and  $r''$  in the PV plane) will define the line ( $r$ ). If the point  $A$  is located in the line  $r$ , its two projections,  $A'$  and  $A''$ , must be in the two projections of the line,  $r'$  and  $r''$  respectively.

The projections of two points located in the line will define the projections of the line.



The intersections of the line with the projection planes (PH and PV) are called traces.

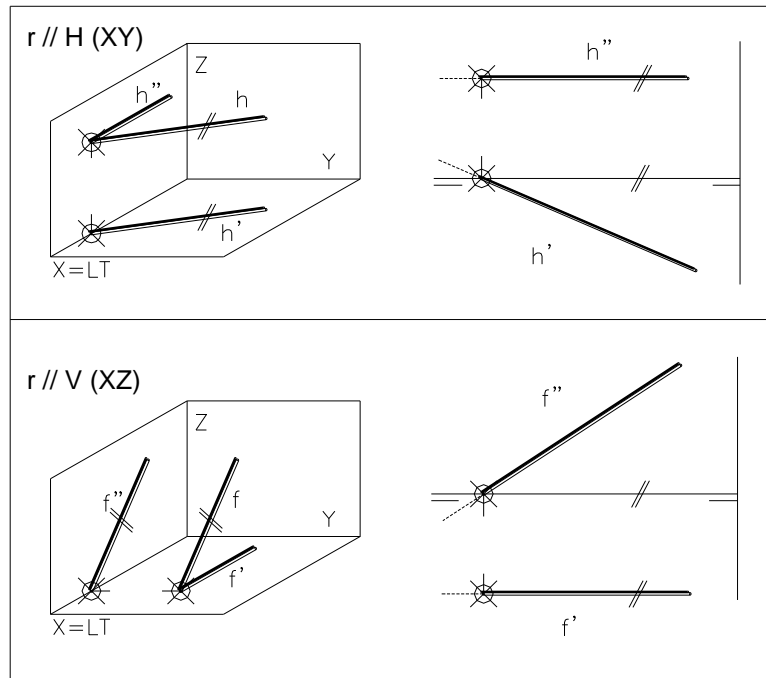
- Horizontal trace (H): the point where the line  $r$  crosses the plane  $XY$  (in this point  $z=0$  is satisfied).
- Vertical trace (V): the point where the line  $r$  crosses the plane  $XZ$  (in this point  $y=0$  is satisfied).



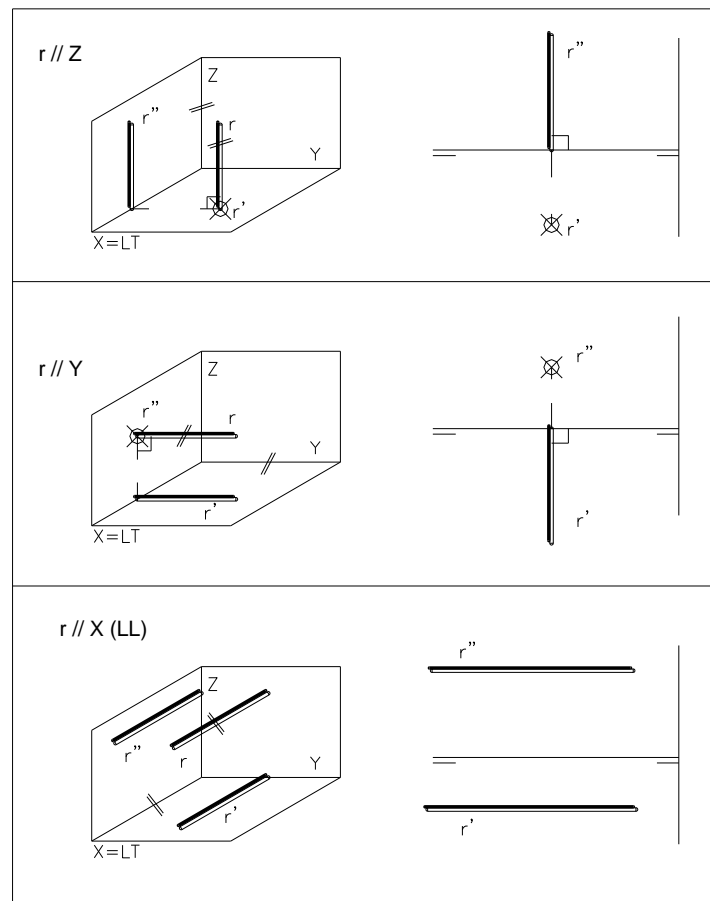
The lines that are parallel to the planes  $XY$  and  $XZ$  have only one trace, because one of its two projections is parallel to the floor-line (LT).

In the next image we can see two of these lines:

- A line that is parallel to  $XY$  (horizontal line):  $h$
- A line that is parallel to  $ZX$  (frontal line):  $f$

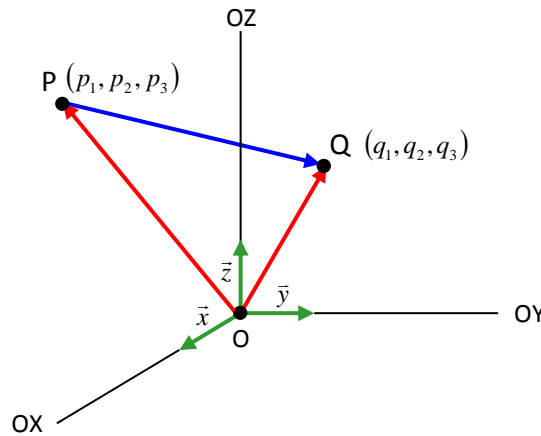


In the next image, parallel lines to the coordinate axes are represented:



### 1.3.A – Coordinates of a vector defined by two points

Let  $P = (p_1, p_2, p_3)$  and  $Q = (q_1, q_2, q_3)$  be two points in the space as in the next image:



The following equality is satisfied  $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$ , therefore:

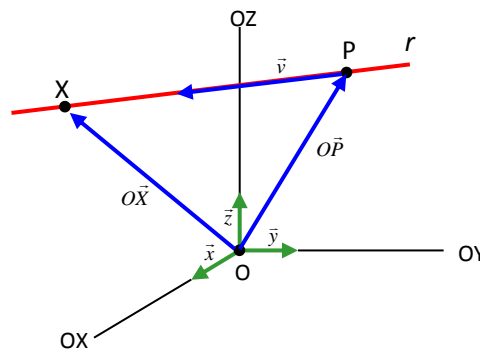
$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} \Rightarrow \overrightarrow{PQ} = (q_1, q_2, q_3) - (p_1, p_2, p_3) = (q_1 - p_1, q_2 - p_2, q_3 - p_3)$$

### 1.3.A - Lines

A point and a direction vector determine a line. A direction vector of the line is a vector with the same direction as the line.

In the reference system  $R = \{O; \vec{x}, \vec{y}, \vec{z}\}$ , the equations of the line  $r$  that passing through the point  $P = (p_1, p_2, p_3)$  is parallel to the vector  $\vec{v} = (v_1, v_2, v_3)$  are the following:

- Vector equation of a line



As it can be seen in the above figure, given any point  $X$  of the line  $r$ , the vectors  $\overrightarrow{PX}$  and  $\vec{v}$  are linearly dependent, thus,  $\overrightarrow{PX} = \lambda \vec{v}$  being  $\lambda \in \mathbb{R}$ . In addition,  $\overrightarrow{OX} = \overrightarrow{OP} + \overrightarrow{PX}$ .

By considering  $\overrightarrow{OX} = \vec{x}$  and  $\overrightarrow{OP} = \vec{p}$ , the expression  $\vec{x} = \vec{p} + \lambda \vec{v}$  is obtained. And this is the vector equation of the line.

- Parametric equations of a line

The parametric equations of a line are obtained by substituting the coordinates of the vectors in the vector equation of the line:

$$r : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ being } \lambda \in \mathbb{R}$$

- Continuous equation of a line

By isolating the parameter  $\lambda$  in the parametric equations, we get:

$$\lambda = \frac{x - p_1}{v_1} ; \lambda = \frac{y - p_2}{v_2} ; \lambda = \frac{z - p_3}{v_3}$$

And making the previous expressions equal, the continuous equation of a line is obtained:

$$r : \frac{x - p_1}{v_1} = \frac{y - p_2}{v_2} = \frac{z - p_3}{v_3}$$

If any of the denominators is zero, the previous expression has symbolic meaning, as the denominators are the coordinates of the direction vector.

- Implicit equations of a line

By considering any two equalities in the continuous equation:

$$\frac{x - p_1}{v_1} = \frac{y - p_2}{v_2} \Rightarrow v_2(x - p_1) = v_1(y - p_2)$$

$$\frac{x - p_1}{v_1} = \frac{z - p_3}{v_3} \Rightarrow v_3(x - p_1) = v_1(z - p_3)$$

The following linear system is obtained:

$$\begin{cases} v_2x - v_1y + (p_2v_1 - p_1v_2) = 0 \\ v_3x - v_1z + (p_3v_1 - p_1v_3) = 0 \end{cases}$$

And if we express the previous system in a general form, the implicit equations of a line are obtained:

$$r : \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

As a particular case, the equations of some lines which are parallel to the coordinate planes or to the coordinate axes are provided:

Equations of the lines that are parallel to the coordinate planes

Line that is parallel to the plane XOY:  $r : \begin{cases} Ax + By + D = 0 \\ z = \text{constant} \end{cases}$



Line that is parallel to the plane XOZ:  $r: \begin{cases} Ax + Cz + D = 0 \\ y = \text{constant} \end{cases}$

Line that is parallel to the plane YOZ:  $r: \begin{cases} By + Cz + D = 0 \\ x = \text{constant} \end{cases}$

### Equations of the lines that are parallel to the coordinate axes

Line that is parallel to the OX axis:  $r: \begin{cases} y = \text{constant} \\ z = \text{constant} \end{cases}$

Line that is parallel to the OY axis:  $r: \begin{cases} x = \text{constant} \\ z = \text{constant} \end{cases}$

Line that is parallel to the OZ axis:  $r: \begin{cases} x = \text{constant} \\ y = \text{constant} \end{cases}$

#### ► **Example 1 (A)**

Obtain the different equations of the line that passes through the points  $A = (5,4,2)$  and  $B = (1,6,4)$ .

*Solution:* The direction vector of the line is  $\vec{v} = B - A = (-4,2,2)$ , therefore:

- Vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix}$$

- Parametric equations

$$\begin{cases} x = 5 - 4\lambda \\ y = 4 + 2\lambda \\ z = 2 + 2\lambda \end{cases}$$

- Continuous equation

$$\frac{x-5}{-4} = \frac{y-4}{2} = \frac{z-2}{2}$$

- Implicit equations

By considering the following two equalities in the continuous equation:

$$\frac{x-5}{-4} = \frac{y-4}{2} \Rightarrow x + 2y - 13 = 0$$

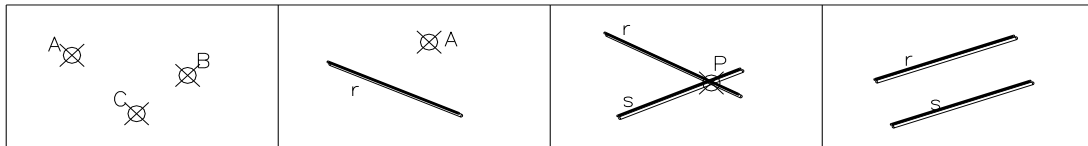
and  $\frac{y-4}{2} = \frac{z-2}{2} \Rightarrow y - z - 2 = 0$

And the implicit equations of the line are given by: 
$$\begin{cases} x + 2y = 13 \\ y - z = 2 \end{cases}$$

### 1.4.G - Plane

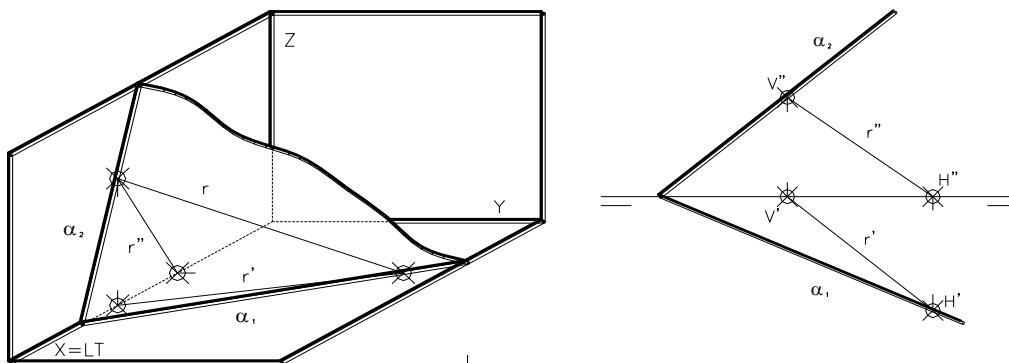
There are different ways to define a plane:

- a) By means of three points that are not aligned;
- b) By means of one line and a point that is out of the line;
- c) By means of two lines that intersect;
- d) By means of two parallel lines.



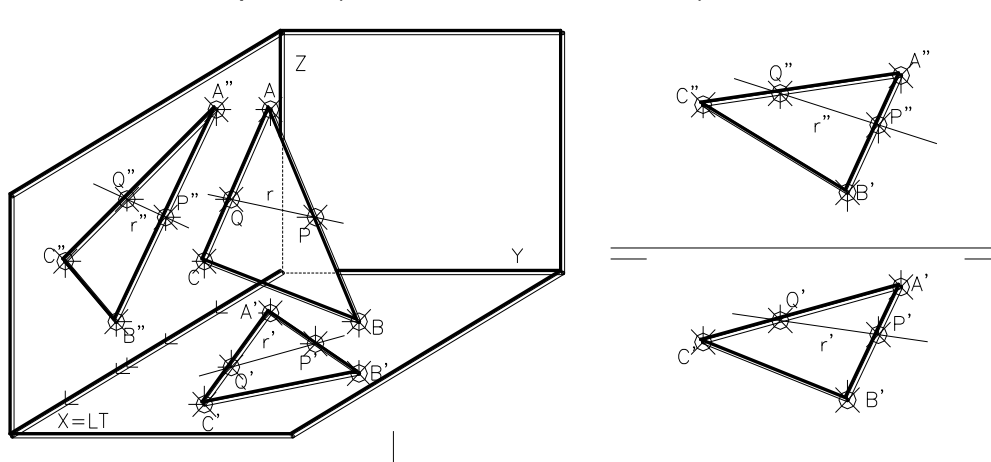
In the diedric system, planes are represented in two ways:

- a. By means of the horizontal and vertical traces of the plane ( $\alpha_1, \alpha_2$ ). These two traces are the intersections (lines) between the plane and the projection-planes XY and XZ.



The traces of the lines that are located in the plane will be in the traces of the plane.

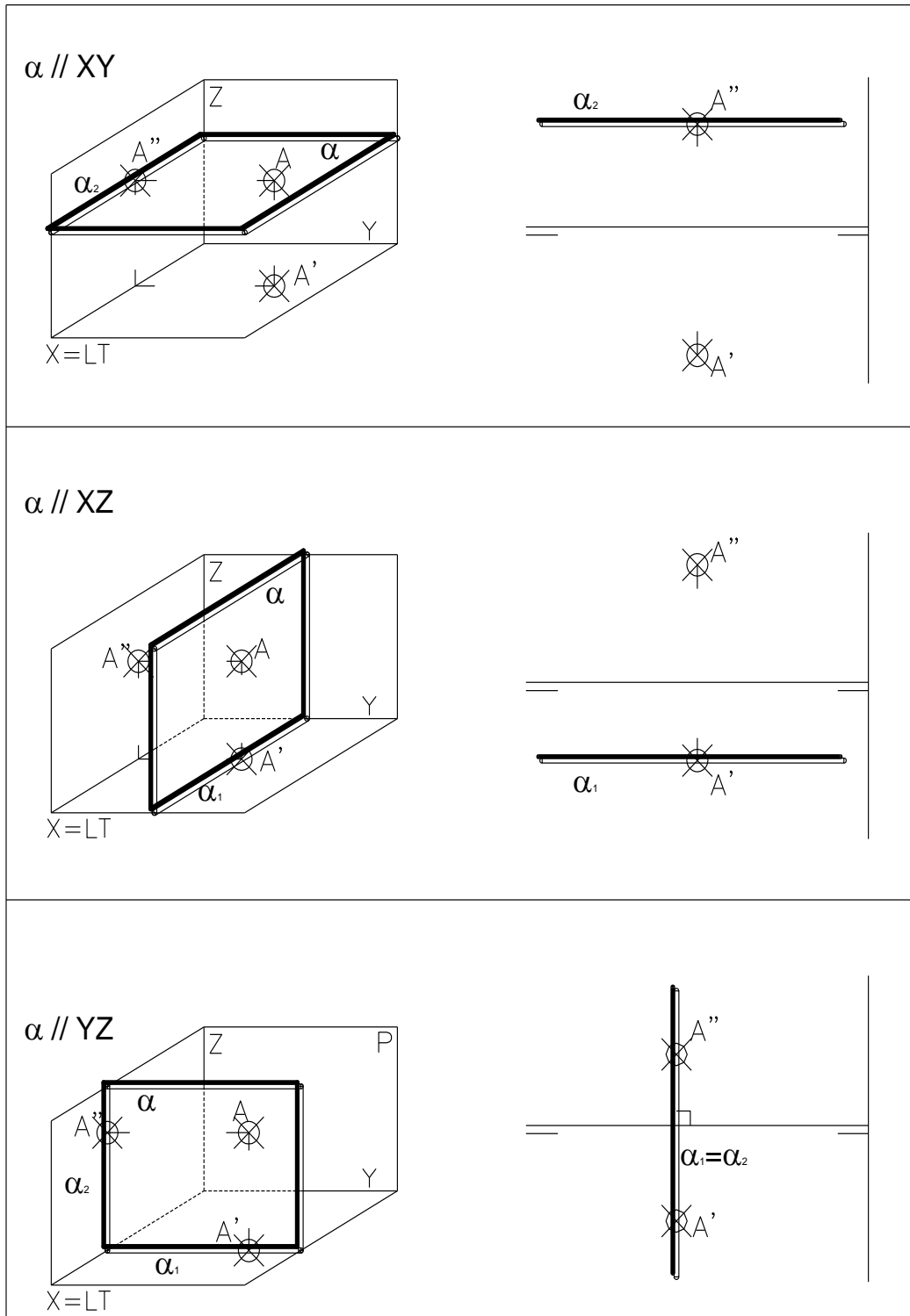
- b. By means of three points (or two lines that intersect):



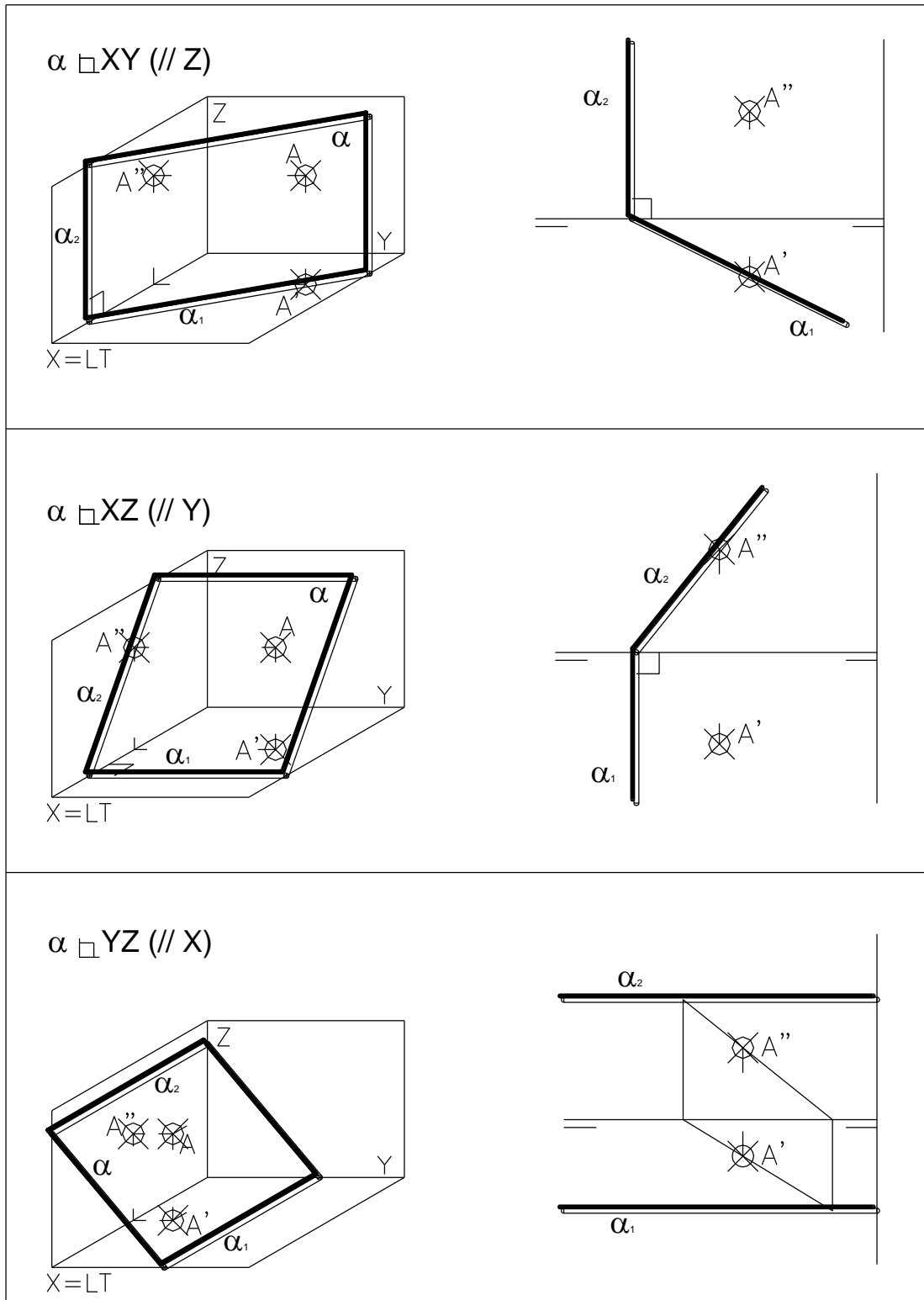
The lines that are located in a plane will intersect at least two lines of the plane.



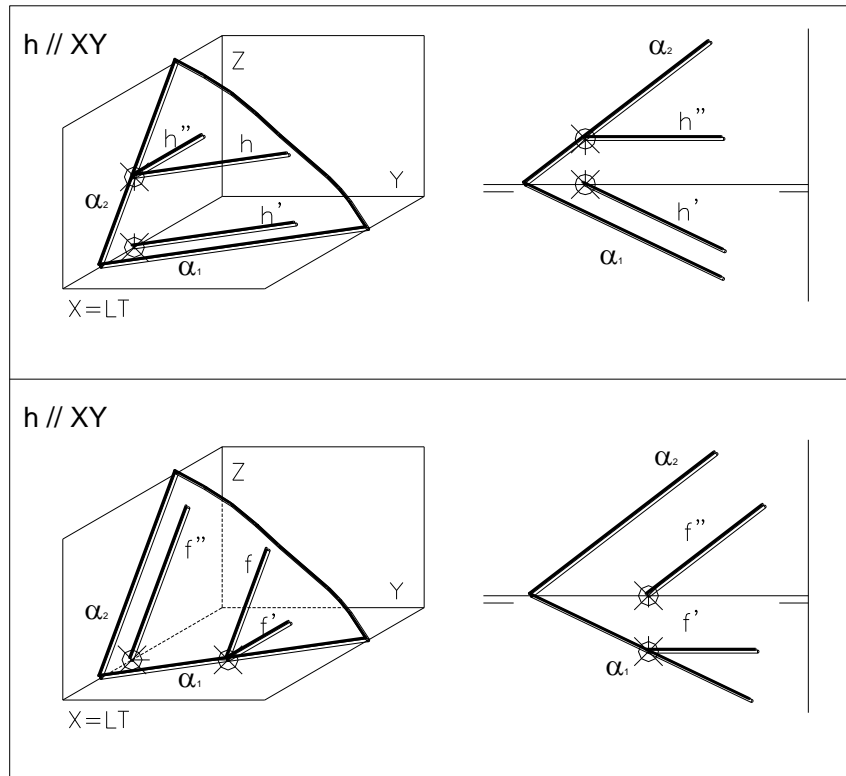
The planes that are parallel to the projection planes have this characteristic: one of the coordinates of all the points in the plane is constant. If the plane is parallel to the PH plane, all the points in the plane will have the same elevation; if the plane is parallel to the PV all the points in the plane will have the same distance to the PV, and if the plane is parallel to the profile-plane (PP) the distance to the PP will be constant.



If a plane is parallel to a coordinate axis, it is perpendicular to a coordinate-plane (will be a perpendicular plane). If the axis is Z, the plane will be perpendicular to the PV; if the axis is Y, it will be perpendicular to the PH; and if the axis is X, it will be perpendicular to the profile-plane (PP). In the next image we can see the three cases.

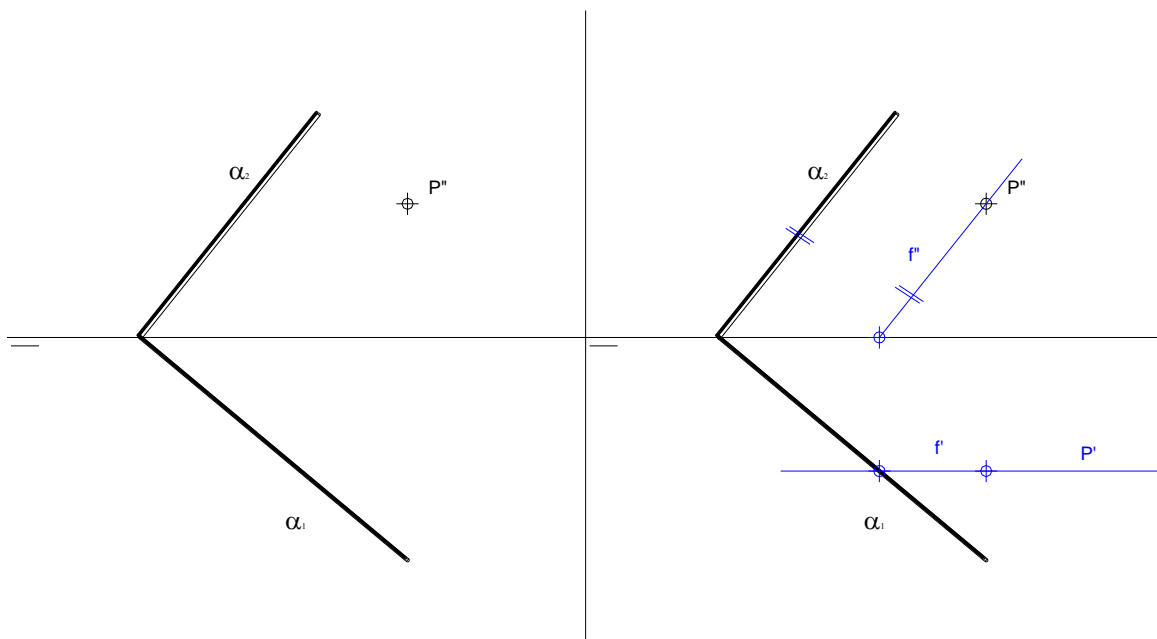


Among the infinite lines included in a plane, some of them are very important to define a plane in the diedric system. For example, the frontal lines (parallel to the PV), and the horizontal lines (parallel to the PH). One of the two projections of these two lines will be parallel to the trace of the plane in which it is located (in the horizontal line  $h$ ,  $h'$  will be // to  $\alpha_1$ ; in the frontal line  $f$ ,  $f''$  will be // to  $\alpha_2$ ).



► **Example 2 (G)**

Draw from the point P a frontal line of the plane. Find the projection of the point that is missing.



### 1.5.A - Plane

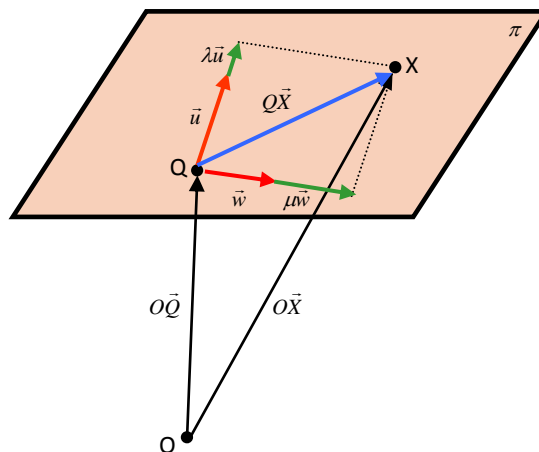
A point and two directions determine a plane. These directions are defined by any two linearly independent vectors included in the plane. These vectors are called direction vectors of the plane.

In the reference system  $R = \{O; \vec{x}, \vec{y}, \vec{z}\}$ , the equations of the plane  $\pi$  that passing through the point  $Q = (q_1, q_2, q_3)$ , has as direction vectors  $\vec{u} = (u_1, u_2, u_3)$  and  $\vec{w} = (w_1, w_2, w_3)$  are the following:

- Vector equation of a plane

As it can be seen in the figure, given any point  $X$  of the plane  $\pi$ , the vector  $\overrightarrow{QX}$  can be written as linear combination of the vectors  $\vec{u}$  and  $\vec{w}$ . Therefore,  $\overrightarrow{QX} = \lambda\vec{u} + \mu\vec{w}$  being  $\lambda, \mu \in \mathbb{R}$ . In addition,  $\overrightarrow{OX} = \overrightarrow{OQ} + \overrightarrow{QX}$ .

By considering  $\overrightarrow{OX} = \vec{x}$  and  $\overrightarrow{OQ} = \vec{q}$ , the expression  $\vec{x} = \vec{q} + \lambda\vec{u} + \mu\vec{w}$  is obtained. And this is the vector equation of the plane.



- Parametric equations of a plane

The parametric equations of a plane are obtained by substituting the coordinates of the vectors in the vector equation of the plane:

$$\pi : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} + \lambda \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \mu \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

- Implicit equation of a plane

By considering the parametric equations as a linear system with two unknowns:

$$\begin{cases} x - q_1 = u_1\lambda + w_1\mu \\ y - q_2 = u_2\lambda + w_2\mu \\ z - q_3 = u_3\lambda + w_3\mu \end{cases}$$

The system will be a determinate compatible system if the following is satisfied:

$$\begin{vmatrix} x - q_1 & y - q_2 & z - q_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} x - q_1 & u_1 & w_1 \\ y - q_2 & u_2 & w_2 \\ z - q_3 & u_3 & w_3 \end{vmatrix} = 0$$

By developing any of the previous determinants, the implicit equation of the plane  $\pi$  is obtained:

$$\pi: Ax + By + Cz + D = 0$$

- Equation of a plane that contains a point

If the associated vector or normal vector of the plane (perpendicular vector to the plane),  $\vec{n} = (A, B, C)$ , and a point  $P = (x_0, y_0, z_0)$  in the plane are known, the equation of the plane is given by:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

As a particular case, the equations of some planes which are parallel to the coordinate planes or to the coordinate axes are provided:

Equations of the planes that are parallel to the coordinate planes

Plane that is parallel to the plane XOY:  $\Rightarrow z = \text{constant}$

Plane that is parallel to the plane XOZ:  $\Rightarrow y = \text{constant}$

Plane that is parallel to the plane YOZ:  $\Rightarrow x = \text{constant}$

Equations of the planes that are parallel to the coordinate axes

Plane that is parallel to the OX axis:  $By + Cz + D = 0$

Plane that is parallel to the OY axis:  $Ax + Cz + D = 0$

Plane that is parallel to the OZ axis:  $Ax + By + D = 0$

► **Example 3 (A)**

Obtain the equations of the plane that contains the points  $A = (5,4,2)$ ,  $B = (1,6,4)$  and  $C = (4,2,5)$ .

*Solution:* The direction vectors of the plane are the following:  $\vec{u} = B - A = (-4,2,2)$  and  $\vec{w} = B - C = (3,-4,1)$ .



- Vector equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 2 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}$$

- Parametric equations

$$\begin{cases} x = 5 - 4\lambda + 3\mu \\ y = 4 + 2\lambda - 4\mu \\ z = 2 + 2\lambda + \mu \end{cases}$$

- Implicit equation

The implicit equation is obtained by solving the following equality:  $\begin{vmatrix} x-5 & y-4 & z-2 \\ -4 & 2 & 2 \\ 3 & -4 & 1 \end{vmatrix} = 0$

Therefore, the implicit equation of the plane is  $x + y + z = 11$ .

- Normal equation

The normal vector of the plane is obtained doing the vector product of the direction vectors:

$$\vec{n} = \vec{v} \wedge \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 2 & 2 \\ 3 & -4 & 1 \end{vmatrix} = 10\vec{i} + 10\vec{j} + 10\vec{k} \Rightarrow \vec{n} = (10, 10, 10)$$

Or we can take  $\vec{n} = (1, 1, 1)$  as the normal vector of the plane, because both vectors have the same direction.

As a point of the plane is known, for example by considering  $A = (5, 4, 2)$ , the normal equation of the plane is obtained:

$$1 \cdot (x - 5) + 1 \cdot (y - 4) + 1 \cdot (z - 2) \Rightarrow x + y + z - 11 = 0$$