Sensitivity analysis. Solutions

1. 1.1 The new optimal solutions:

   1.1.1 $x_1^* = 0, \quad x_2^* = 0, \quad x_3^* = 2, \quad z^* = 10.$

   1.1.2 $x_1^* = 2, \quad x_2^* = 0, \quad x_3^* = 2, \quad z^* = 18.$

   1.1.3 $x_1^* = 0, \quad x_2^* = 1, \quad x_3^* = 3, \quad z^* = 18.$

   1.1.4 $x_1^* = 2, \quad x_2^* = 0, \quad x_3^* = 2, \quad z^* = 24.$

   1.1.5 $x_1^* = 2, \quad x_2^* = 0, \quad x_3^* = 2, \quad z^* = 18.$

   1.1.6 Multiple optimal solutions, $z^* = 18$.

   1.1.7 $x_1^* = 2, \quad x_2^* = 0, \quad x_3^* = 1, \quad x_4^* = 1, \quad z^* = 19.$

   1.1.8 $x_1^* = 2, \quad x_2^* = 0, \quad x_3^* = 2, \quad x_4^* = 0, \quad z^* = 18.$

   1.1.9 $x_1^* = 2, \quad x_2^* = 0, \quad x_3^* = 2, \quad z^* = 18.$

   1.1.10 $x_1^* = \frac{1}{2}, \quad x_2^* = 0, \quad x_3^* = 3, \quad z^* = 17.$

   1.2 The interpretation of shadow prices.

   - The interpretation of the shadow price for the first resource, b_1.

     $$\overset{\wedge}{\mathbf{b}} = \begin{pmatrix} 5 \\ 10 \\ 16 \end{pmatrix}, \quad \mathbf{x}_B = \mathbf{B}^{-1}\overset{\wedge}{\mathbf{b}} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} \geq \mathbf{0}.$$

     y_1^* is the shadow price for b_1. If b_1 changes from 4 to 5, the new optimal objective value is $\hat{z} = z + y_1^* = 18 + 2 = 20$, and the new optimal solution $x_1^* = 5, \quad x_2^* = 0, \quad x_3^* = 0.$

   - The interpretation of the shadow price for the second resource, b_2.

     $$\overset{\wedge}{\mathbf{b}} = \begin{pmatrix} 4 \\ 11 \\ 16 \end{pmatrix}, \quad \mathbf{x}_B = \mathbf{B}^{-1}\overset{\wedge}{\mathbf{b}} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \geq \mathbf{0}.$$

     y_2^* is the shadow price for b_2. If b_2 changes from 10 to 11, the new optimal objective value is $\hat{z} = z + y_2^* = 18 + 1 = 19$, and the new optimal solution $x_1^* = 1, \quad x_2^* = 0, \quad x_3^* = 3.$

   - The interpretation of the shadow price for the third resource, b_3.

     $$\overset{\wedge}{\mathbf{b}} = \begin{pmatrix} 4 \\ 10 \\ 15 \end{pmatrix}, \quad \mathbf{x}_B = \mathbf{B}^{-1}\overset{\wedge}{\mathbf{b}} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \geq \mathbf{0}.$$

     y_3^* is the shadow price for b_3. If b_3 changes from 16 to 15, the new optimal objective value is $\hat{z} = z - y_3^* = 18 - 0 = 18$, and the new optimal solution $x_1^* = 2, \quad x_2^* = 0, \quad x_3^* = 2.$

1.3 The range of values that leaves the current basis unchanged, for each of the components in vector $\mathbf{c}$:

$$\frac{10}{3} \leq c_1 \leq 5, \qquad c_2 \leq 3, \qquad 4 \leq c_3 \leq 6.$$

The range of values that leaves the current basis unchanged, for each of the components in vector $\mathbf{b}$:

$$\frac{10}{3} \leq b_1 \leq 5, \qquad 8 \leq b_2 \leq 12, \qquad b_3 \geq 14.$$

2. 2.1 The new optimal solutions:

2.1.1 $x_1^* = 5, \quad x_2^* = 1, \quad x_3^* = 0, \quad z^* = 26.$

2.1.2 $x_1^* = 8, \quad x_2^* = 0, \quad x_3^* = 1, \quad z^* = 37.$

2.1.3 $x_1^* = 5, \quad x_2^* = 1, \quad x_3^* = 0, \quad z^* = 38.$

2.1.4 $x_1^* = 0, \quad x_2^* = \frac{8}{3}, \quad x_3^* = \frac{5}{3}, \quad z^* = 31.$

2.1.5 $x_1^* = 5, \quad x_2^* = 1, \quad x_3^* = 0, \quad z^* = 26.$

2.1.6 $x_1^* = 0, \quad x_2^* = 1, \quad x_3^* = 5, \quad z^* = 31.$

2.1.7 $x_1^* = 5, \quad x_2^* = 1, \quad x_3^* = 0, \quad x_4^* = 0, \quad z^* = 26.$

2.1.8 $x_1^* = 6, \quad x_2^* = 0, \quad x_3^* = 0, \quad x_4^* = 3, \quad z^* = 39.$

2.1.9 $x_1^* = 2, \quad x_2^* = 2, \quad x_3^* = 1, \quad z^* = 25.$

2.1.10 $x_1^* = 5, \quad x_2^* = 1, \quad x_3^* = 0, \quad z^* = 26.$

2.2 The interpretation of shadow prices.

- The interpretation of the shadow price for the first resource, b_1.

$$\overset{\wedge}{\mathbf{b}} = \begin{pmatrix} 11 \\ 14 \\ 6 \end{pmatrix}, \quad \mathbf{x}_B = \mathbf{B}^{-1} \overset{\wedge}{\mathbf{b}} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} \geq \mathbf{0}.$$

  y_1^* is the shadow price for b_1. If b_1 changes from 12 to 11, the new optimal objective value is $\hat{z} = z - y_1^* = 26 - 0 = 26$, and the new optimal solution $x_1^* = 5, \quad x_2^* = 1, \quad x_3^* = 0.$

- The interpretation of the shadow price for the second resource, b_2.

$$\overset{\wedge}{\mathbf{b}} = \begin{pmatrix} 12 \\ 15 \\ 6 \end{pmatrix}, \quad \mathbf{x}_B = \mathbf{B}^{-1} \overset{\wedge}{\mathbf{b}} = \begin{pmatrix} \frac{9}{2} \\ \frac{3}{2} \\ \frac{9}{2} \end{pmatrix} \geq \mathbf{0}.$$

  y_2^* is the shadow price for b_2. If b_2 changes from 14 to 15, the new optimal objective value is $\hat{z} = z + y_2^* = 26 + 1 = 27$, and the new optimal solution $x_1^* = \frac{9}{2}, \quad x_2^* = \frac{3}{2}, \quad x_3^* = 0.$

- The interpretation of the shadow price for the third resource, b_3.

$$\overset{\wedge}{\mathbf{b}}= \begin{pmatrix} 12 \\ 14 \\ 7 \end{pmatrix}, \quad \mathbf{x}_B = \mathbf{B}^{-1} \overset{\wedge}{\mathbf{b}}= \begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix} \geq \mathbf{0}.$$

  y_3^* is the shadow price for b_3. If b_3 changes from 6 to 7, the new optimal objective value is $\hat{z}= z + y_3^* = 26 + 2 = 28$, and the new optimal solution $x_1^* = 7, \quad x_2^* = 0, \quad x_3^* = 0$.

2.3 The range of values that leaves the current basis unchanged, for each of the components in vector $\mathbf{c}$:

$$\frac{11}{3} \leq c_1 \leq 6, \qquad 4 \leq c_2 \leq 7, \qquad c_3 \leq 6.$$

The range of values that leaves the current basis unchanged, for each of the components in vector $\mathbf{b}$:

$$b_1 \geq 7, \qquad 12 \leq b_2 \leq 24, \qquad \frac{7}{2} \leq b_3 \leq 7.$$

3. 3.1 8 cookery books of type B_1 will be designed, 4 cookery books of type B_2, and 2 cookery books of type B_3:

$$x_1^* = 8, \quad x_2^* = 4, \quad x_3^* = 2.$$

A total amount of $z^* = 14$ cookery book designs will be produced.

3.2 The optimal solution $x_1^* = 8,\ x_2^* = 4,\ x_3^* = 2$ verifies the three constraints with equality. Therefore, all the cooks take part in a team.

3.3 If $c_1 = 2$, the optimal solution is $x_1^* = 8,\ x_2^* = 4,\ x_3^* = 2, z^* = 22$. If $c_1 = 3$, there are multiple optimal solutions.

$$x_1^* = 8,\ x_2^* = 4,\ x_3^* = 2, z^* = 30.$$

$$x_1^* = 10,\ x_2^* = 0,\ x_3^* = 0, z^* = 30.$$

c_1 can be increased $c_1 \leq 3$ at most, if we want the number of type B_i cookery books designed to remain unchanged.

3.4 10 cookery books of type B_1 will be designed, 5 cookery books of type B_2 and 0 cookery books of type B_3:

$$x_1^* = 10, \quad x_2^* = 5, \quad x_3^* = 0.$$

Thus, no cookery book of type B_3 will be produced. A total amount of $z^* = 15$ cookery books will be designed.

4. 4.1 The interpretation of shadow prices.

- The interpretation of the shadow price for the first resource b_1, (red paint).

$$\stackrel{\wedge}{\mathbf{b}}= \begin{pmatrix} 25 \\ 14 \\ 32 \end{pmatrix}, \quad \mathbf{x}_B = \mathbf{B}^{-1} \stackrel{\wedge}{\mathbf{b}} \geq \mathbf{0}.$$

y_1^* is the shadow price for b_1, and since $y_1^* = 0$, if 25 kg of red paint were bought instead of 26 kg, the optimal benefit would remain the same, $\stackrel{\wedge}{z}= z - y_1^* = 224 - 0 = 224$. By substituting the optimal solution into the first constraint we can see that too much red paint is being bought, because only 21 kg are being used.

- The interpretation of the shadow price for the second resource b_2, (blue paint).

$$\stackrel{\wedge}{\mathbf{b}}= \begin{pmatrix} 26 \\ 15 \\ 32 \end{pmatrix}, \quad \mathbf{x}_B = \mathbf{B}^{-1} \stackrel{\wedge}{\mathbf{b}} \geq \mathbf{0}.$$

y_2^* is the shadow price for b_2, and since $y_2^* = 16$, if one more kg of blue paint was bought (15 kg instead of 14 kg), the optimal benefit would be increased by $\stackrel{\wedge}{z}= z + y_2^* = 224 + 16 = 240$. By substituting the optimal solution into the second constraint we can see that all the blue paint available is currently being used in the production of the new colors.

- The interpretation of the shadow price for the third resource b_3, (yellow paint).

$$\stackrel{\wedge}{\mathbf{b}}= \begin{pmatrix} 26 \\ 14 \\ 31 \end{pmatrix}, \quad \mathbf{x}_B = \mathbf{B}^{-1} \stackrel{\wedge}{\mathbf{b}} \geq \mathbf{0}.$$

y_3^* is the shadow price for b_3, and since $y_3^* = 0$, if 31 kg of yellow paint were bought instead of 32 kg, the optimal benefit would remain the same $\stackrel{\wedge}{z}= z - y_3^* = 224 - 0 = 224$. By substituting the optimal solution into the third constraint we can see that too much yellow paint is being bought; there is more yellow paint than required.

4.2 At most $\frac{52}{3}$ kg can be bought.

4.3 The new optimal solution:

$$x_1^* = 0, \quad x_2^* = 48, \quad x_3^* = 0, \quad x_4^* = 20, \quad z^* = 312.$$