Duality. Solutions

1. The dual models to the given linear problems.

1.1.
$$\max\ G = y_1 + 7y_2 + 10y_3$$
subject to
$$\begin{aligned} y_1 + 2y_2 + y_3 &\geq 2 \\ 2y_1 - 2y_2 + 2y_3 &\leq 3 \\ 5y_1 + 4y_2 + y_3 &= -4 \\ y_1, y_3 \geq 0, y_2 &: \text{unrestricted} \end{aligned}$$

1.2.
$$\max\ G = -7y_1 + 12y_2 + 5y_3$$
subject to
$$\begin{aligned} 4y_1 + 2y_2 + 2y_3 &\leq 1 \\ -y_1 - 4y_2 + 8y_3 &\leq 3 \\ 2y_1 + 4y_3 &\leq 1 \\ y_1 \leq 0, y_2, y_3 &\geq 0 \end{aligned}$$

1.3.
$$\min\ G = 12y_1 - 8y_2 + 10y_3$$
subject to
$$\begin{aligned} 2y_1 - y_2 + 3y_3 &\leq 2 \\ y_1 + 5y_2 + 4y_3 &\geq 2 \\ 2y_1 - 2y_2 - 6y_3 &\geq 5 \\ y_1 : \text{unrestricted}, y_2 \leq 0, y_3 &\geq 0 \end{aligned}$$

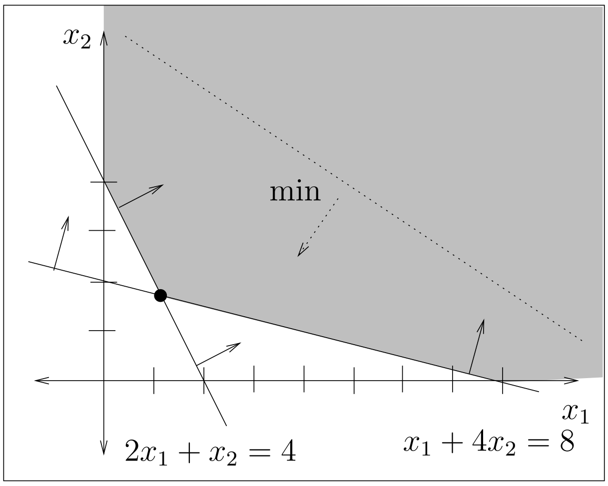
1.4.
$$\min\ G = -4y_1 + 2y_2 + 6y_3$$
subject to
$$\begin{aligned} y_1 - y_2 + 4y_3 &\geq 1 \\ y_1 + 6y_2 - y_3 &\geq 1 \\ 2y_1 + 2y_2 + y_3 &= 5 \\ y_1 \geq 0, y_2 \leq 0, y_3 &: \text{unrestricted} \end{aligned}$$

1.5.
$$\max\ G = -6y_1 + 6y_2 + 10y_3$$
subject to
$$\begin{aligned} 4y_1 + y_2 + 5y_3 &\geq 4 \\ -2y_1 + y_2 + 2y_3 &\geq 1 \\ 3y_1 + y_2 - y_3 &\leq -1 \\ y_1 + y_2 - y_3 &\leq 2 \\ y_1 \leq 0, y_2 : \text{unrestricted}, y_3 &\geq 0 \end{aligned}$$

1.6.
$$\min\ G = 14y_1 - 6y_2 + 10y_3 + 3y_4$$
subject to
$$\begin{aligned} 2y_1 - y_2 + 4y_3 + y_4 &\geq 1 \\ -4y_1 + 8y_2 + 6y_3 + 9y_4 &\leq 4 \\ y_1, y_3 \geq 0, y_2 \leq 0, y_4 &: \text{unrestricted} \end{aligned}$$

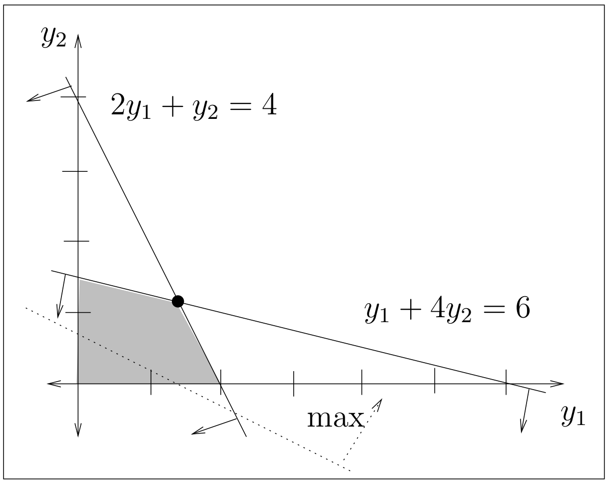
2. Graphical solutions for the given models and the associated dual models.

   2.1 Unique optimal solution, $x_1^* = \frac{8}{7},\ x_2^* = \frac{12}{7},\ z^* = \frac{104}{7}$.

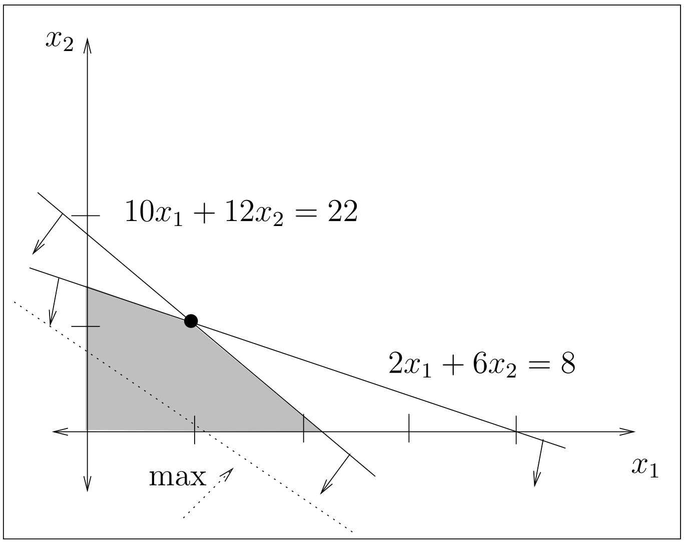


$$
\begin{aligned}
\max\ G &= 4y_1 + 8y_2 \\
\text{subject to} & \\
2y_1 + y_2 &\leq 4 \\
y_1 + 4y_2 &\leq 6 \\
y_1, y_2 &\geq 0
\end{aligned}
$$

The dual model has a unique optimal solution, $y_1^* = \frac{10}{7},\ y_2^* = \frac{8}{7},\ G^* = \frac{104}{7}$.

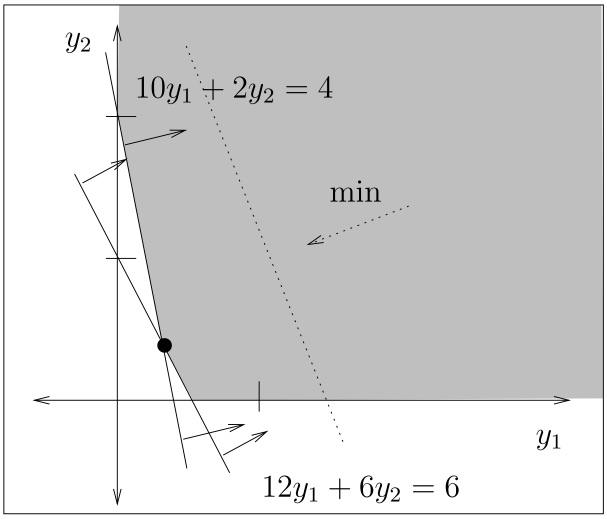


2.2 Unique optimal solution, $x_1^* = 1,\ x_2^* = 1,\ z^* = 10.$



$$
\begin{aligned}
\min\ G = 22y_1 + 8y_2 \\
\text{subject to} \\
10y_1 + 2y_2 \geq 4 \\
12y_1 + 6y_2 \geq 6 \\
y_1, y_2 \geq 0
\end{aligned}
$$

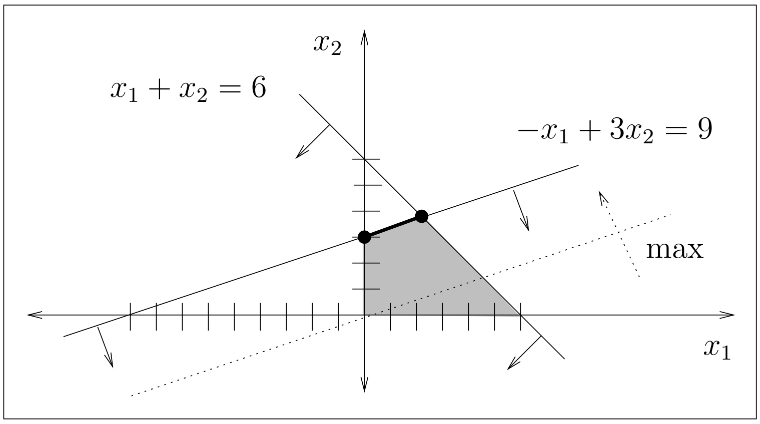
The dual model has a unique optimal solution, $y_1^* = \frac{1}{3},\ y_2^* = \frac{1}{3},\ G^* = 10.$



2.3 Multiple optimal solutions; the two extreme points

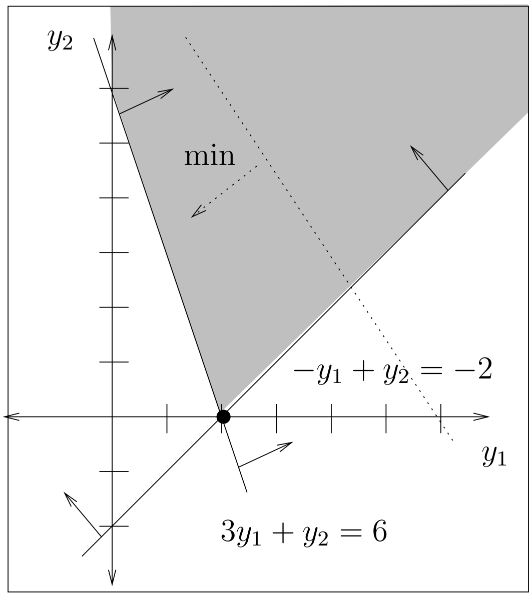
$$x_1^* = 0,\ x_2^* = 3 \quad \text{and} \quad x_1^* = \frac{9}{4},\ x_2^* = \frac{15}{4},$$

and the infinite points lying on the segment line. $z^* = 18$ for all of them.

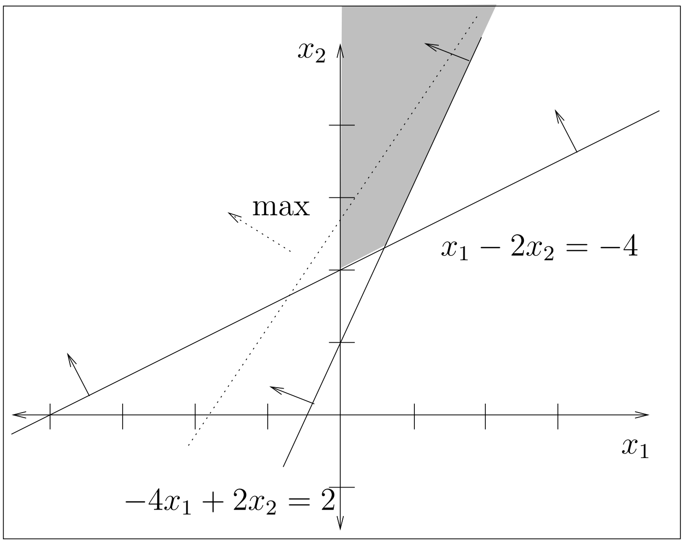


$$\begin{aligned} \min\ G &= 9y_1 + 6y_2 \\ \text{subject to} & \\ -y_1 + y_2 &\geq -2 \\ 3y_1 + y_2 &\geq 6 \\ y_1, y_2 &\geq 0 \end{aligned}$$

The dual model has a unique optimal solution, $y_1^* = 2,\ y_2^* = 0,\ G^* = 18$.

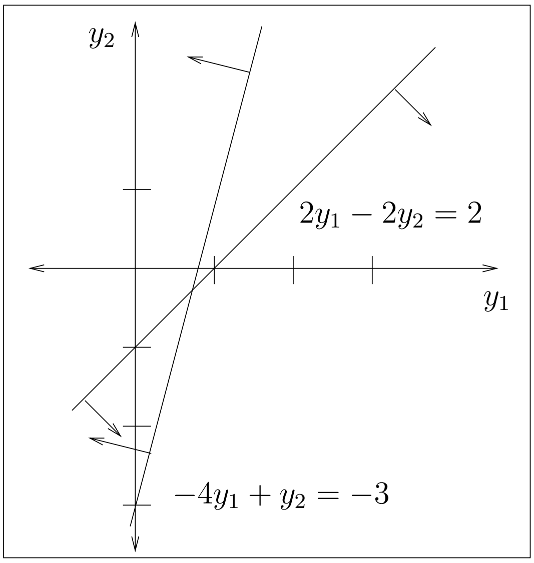


2.4 The problem is unbounded.



$$
\begin{aligned}
\min \; G = 2y_1 - 4y_2 \\
\text{subject to} \\
-4y_1 + y_2 \geq -3 \\
2y_1 - 2y_2 \geq 2 \\
y_1 \leq 0, y_2 \geq 0
\end{aligned}
$$

The dual problem is infeasible.



3. The solution to the linear models applying the dual simplex algorithm.

3.1 A unique optimal solution.

$$x_1^* = 4, \quad x_2^* = 0, \quad x_3^* = 0, \quad z^* = -8$$

3.2 A unique optimal solution.

$$x_1^* = 0, \quad x_2^* = 3, \quad x_3^* = 8, \quad x_4^* = 0, \quad z^* = 27$$

3.3 Multiple optimal solutions, $z^* = -30$.

3.4 The problem is infeasible.

3.5 The problem is infeasible.

3.6 A unique optimal solution.

$$x_1^* = 0, \quad x_2^* = 6, \quad x_3^* = 0, \quad x_4^* = 12, \quad z^* = 84$$

3.7 A unique optimal solution.

$$x_1^* = 11, \quad x_2^* = 0, \quad x_3^* = 1, \quad x_4^* = 0, \quad z^* = 35$$

3.8 The problem is unbounded.

3.9 The problem is unbounded.

4. 4.1 The dual model:

$$\begin{aligned} \min\ G = 2y_1 + 3y_2 + 3y_3 + 2y_4 \\ \text{subject to} \\ y_1 + 2y_2 + 2y_3 + 4y_4 \geq 10 \\ 2y_1 + y_2 + 2y_3 + y_4 \geq 6 \\ y_1, y_2, y_3, y_4 \geq 0 \end{aligned}$$

4.2 The optimal solution to the dual model obtained applying the dual simplex algorithm:

$$y_1^* = 2, \quad y_2^* = 0, \quad y_3^* = 0, \quad y_4^* = 2, \quad G^* = 8.$$

4.3 The optimal solution to the primal problem extracted directly from the optimal tableau computed for the dual problem:

$$x_1^* = \frac{2}{7}, \quad x_2^* = \frac{6}{7}, \quad z^* = 8.$$

5. 5.1 The dual model:

$$\begin{aligned} \max\ G = 20y_1 + 16y_2 + 18y_3 + 21y_4 \\ \text{subject to} \\ 4y_1 + 6y_2 + 4y_3 + 4y_4 \leq 30 \\ 2y_1 + 4y_2 + 2y_3 + 4y_4 \leq 28 \\ y_1, y_2, y_3, y_4 \geq 0 \end{aligned}$$

5.2 The optimal solution to the dual model obtained applying the simplex algorithm:

$$y_1^* = 1, \quad y_2^* = 0, \quad y_3^* = 0, \quad y_4^* = \frac{13}{2}, \quad G^* = \frac{313}{2}.$$

5.3 The optimal solution to the primal problem extracted directly from the optimal tableau computed for the dual problem:

$$x_1^* = \frac{19}{4}, \quad x_2^* = \frac{1}{2}, \quad z^* = \frac{313}{2}.$$

6. 6.1 (a) The optimal solution to the model:

$$x_1^* = \frac{7}{2}, \quad x_2^* = \frac{3}{2}, \quad x_3^* = 0, \quad z^* = \frac{57}{2}.$$

(b) The optimal solution to the dual model is $y_1^* = \frac{3}{20}, \quad y_2^* = \frac{1}{4}, \quad G^* = \frac{57}{2}$. The dual model:

$$\begin{aligned} \min \; G = 90y_1 + 60y_2 \\ \text{subject to} \\ 15y_1 + 15y_2 \geq 6 \\ 25y_1 + 5y_2 \geq 5 \\ 30y_1 + 15y_2 \geq 4 \\ y_1, y_2 \geq 0 \end{aligned}$$

(c) The interpretation of the shadow price associated with the resource b_1.

$$\overset{\wedge}{\mathbf{b}} = \begin{pmatrix} 91 \\ 60 \end{pmatrix}, \quad \mathbf{x}_B = \mathbf{B}^{-1} \overset{\wedge}{\mathbf{b}} = \begin{pmatrix} \frac{31}{20} \\ \frac{209}{60} \end{pmatrix} \geq \mathbf{0}$$

y_1^* is the shadow price associated with the resource b_1. If the resource b_1 increases from 90 units to 91 units, $\hat{z} = z + y_1^* = \frac{57}{2} + \frac{3}{20} = \frac{573}{20}$ holds, and the new optimal solution:

$$x_1^* = \frac{209}{60}, \quad x_2^* = \frac{31}{20}, \quad x_3^* = 0$$

The interpretation of the shadow price associated with the resource b_2.

$$\overset{\wedge}{\mathbf{b}} = \begin{pmatrix} 90 \\ 61 \end{pmatrix}, \quad \mathbf{x}_B = \mathbf{B}^{-1} \overset{\wedge}{\mathbf{b}} = \begin{pmatrix} \frac{29}{20} \\ \frac{215}{60} \end{pmatrix} \geq \mathbf{0}.$$

y_2^* is the shadow price associated with the resource b_2. If the resource b_2 increases from 60 units to 61 units, $\hat{z} = z + y_2^* = \frac{57}{2} + \frac{1}{4} = \frac{115}{4}$ holds, and the new optimal solution:

$$x_1^* = \frac{215}{60}, \quad x_2^* = \frac{29}{20}, \quad x_3^* = 0.$$

6.2 (a) The optimal solution to the model:

$$x_1^* = 12, \quad x_2^* = 0, \quad x_3^* = 0, \quad z^* = 24.$$

(b) The optimal solution to the dual model is $y_1^* = 2,\ \ y_2^* = 0,\ \ G^* = 24$. The dual model:

$$\begin{aligned} \min\ G &= 12y_1 + 8y_2 \\ \text{subject to} & \\ y_1 + 4y_2 &\geq 2 \\ 2y_1 + 2y_2 &\geq 1 \\ 4y_1 &\geq -1 \\ y_1 \geq 0, y_2 &\leq 0 \end{aligned}$$

(c) The interpretation of the shadow price associated with the resource b_1.

$$\overset{\wedge}{\mathbf{b}}= \begin{pmatrix} 13 \\ 8 \end{pmatrix}, \quad \mathbf{x}_B = \mathbf{B}^{-1} \overset{\wedge}{\mathbf{b}}= \begin{pmatrix} 44 \\ 13 \end{pmatrix} \geq \mathbf{0}.$$

y_1^* is the shadow price associated with the resource b_1. If the resource b_1 increases from 12 units to 13 units, $\hat{z}= z + y_1^* = 24 + 2 = 26$ holds, and the new optimal solution:

$$x_1^* = 13, \quad x_2^* = 0, \quad x_3^* = 0.$$

The interpretation of the shadow price associated with the resource b_2.

$$\overset{\wedge}{\mathbf{b}}= \begin{pmatrix} 12 \\ 7 \end{pmatrix}, \quad \mathbf{x}_B = \mathbf{B}^{-1} \overset{\wedge}{\mathbf{b}}= \begin{pmatrix} 41 \\ 12 \end{pmatrix} \geq \mathbf{0}.$$

y_2^* is the shadow price associated with the resource b_2. If the resource b_2 decreases from 8 units to 7 units, $\hat{z}= z - y_2^* = 24 - 0 = 24$ holds, and the new optimal solution:

$$x_1^* = 12, \quad x_2^* = 0, \quad x_3^* = 0.$$