The Simplex Method. Solutions

1. The following are the maximization standard forms of the models:

1.1

$$\begin{aligned}
\max z = 2x_1 + 4x_2 - 4x_3' + 4x_3'' + 0x_4 + 0x_5 \\
\text{subject to} \\
3x_1 + 2x_2 + 4x_3' - 4x_3'' - x_4 = 1 \\
4x_1 - 3x_2 = 2 \\
2x_1 + x_2 + 6x_3' - 6x_3'' + x_5 = 3 \\
x_1, x_2, x_3', x_3'', x_4, x_5 \geq 0, x_3 = x_3' - x_3''
\end{aligned}$$

1.2

$$\begin{aligned}
\max (-z) = -2x_1 + 3x_2 - x_3 + 0x_4 + 0x_5 \\
\text{subject to} \\
x_1 - 5x_2 + 6x_3 - x_4 = 8 \\
-x_1 + 4x_2 - x_5 = 12 \\
2x_1 - x_2 + 4x_3 = 5 \\
x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}$$

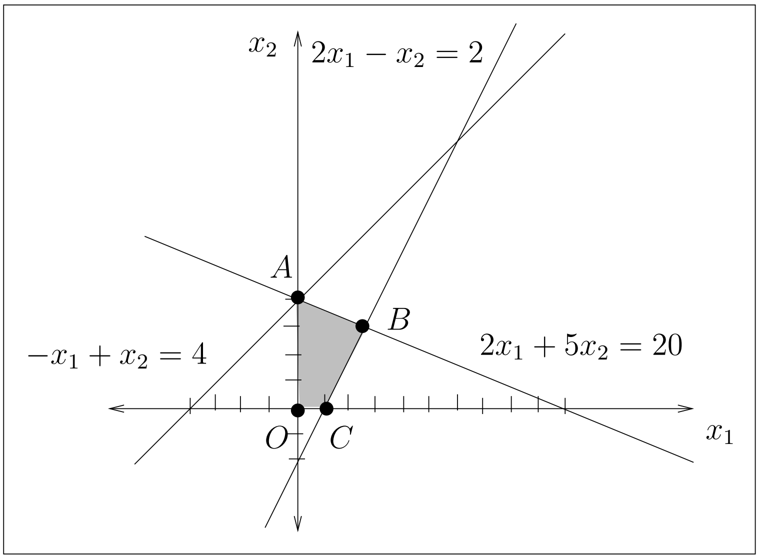
1.3

$$\begin{aligned}
\max (-z) = 2x_1' - 2x_2 + 4x_3 + 0x_4 + 0x_5 \\
\text{subject to} \\
-2x_1' + 2x_2 + 2x_3 = 10 \\
-2x_1' - 6x_2 + x_3 - x_4 = 10 \\
x_1' + 3x_2 - x_5 = 3 \\
x_1', x_2, x_3, x_4, x_5 \geq 0, x_1' = -x_1
\end{aligned}$$

1.4

$$\begin{aligned}
\max z = -3x_1' - 7x_2 + 5x_3' - 5x_3'' + 0x_4 + 0x_5 \\
\text{subject to} \\
-x_2 + x_3' - x_3'' - x_4 = 9 \\
x_1' - 2x_3' + 2x_3'' - x_5 = 5 \\
-4x_1' - x_2 = 6 \\
x_1', x_2, x_3', x_3'', x_4, x_5 \geq 0 \\
x_1' = -x_1, x_3 = x_3' - x_3''
\end{aligned}$$

2. Basic solutions and their corresponding extreme points in the graphical representation.



There are 10 basic solutions.

- $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3), \mathbf{x}_B = \begin{pmatrix} \frac{5}{2} \\ 3 \\ \frac{7}{2} \end{pmatrix} \geq \mathbf{0}, \quad x_1 = \frac{5}{2}, x_2 = 3 \quad \rightarrow \quad B$

- $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_4), \mathbf{x}_B = \begin{pmatrix} 6 \\ 10 \\ -42 \end{pmatrix} \not\geq \mathbf{0}, \quad x_1 = 6, x_2 = 10$

- $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_5), \mathbf{x}_B = \begin{pmatrix} 0 \\ 4 \\ 6 \end{pmatrix} \geq \mathbf{0}, \quad x_1 = 0, x_2 = 4 \quad \rightarrow \quad A$ (degenerate)

- $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_3\ \mathbf{a}_4), \mathbf{x}_B = \begin{pmatrix} 1 \\ 5 \\ 18 \end{pmatrix} \geq \mathbf{0}, \quad x_1 = 1, x_2 = 0 \quad \rightarrow \quad C$

- $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_3\ \mathbf{a}_5), \mathbf{x}_B = \begin{pmatrix} 10 \\ 14 \\ -18 \end{pmatrix} \not\geq \mathbf{0}, \quad x_1 = 10, x_2 = 0$

- $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_4\ \mathbf{a}_5), \mathbf{x}_B = \begin{pmatrix} -4 \\ 28 \\ 10 \end{pmatrix} \not\geq \mathbf{0}, \quad x_1 = -4, x_2 = 0$

- $\mathbf{B} = (\mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4), \mathbf{x}_B = \begin{pmatrix} -2 \\ 6 \\ 30 \end{pmatrix} \not\geq \mathbf{0}, \quad x_1 = 0, x_2 = -2$

- $\mathbf{B} = (\mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_5), \mathbf{x}_B = \begin{pmatrix} 4 \\ 0 \\ 6 \end{pmatrix} \geq \mathbf{0}, \quad x_1 = 0, x_2 = 4 \quad \rightarrow \quad A$ (degenerate)

- $\mathbf{B} = (\mathbf{a}_2\ \mathbf{a}_4\ \mathbf{a}_5), \mathbf{x}_B = \begin{pmatrix} 4 \\ 0 \\ 6 \end{pmatrix} \geq \mathbf{0}, \quad x_1 = 0, x_2 = 4 \quad \rightarrow \quad A$ (degenerate)

- $\mathbf{B} = (\mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5), \mathbf{x}_B = \begin{pmatrix} 4 \\ 20 \\ 2 \end{pmatrix} \geq \mathbf{0}, \quad x_1 = 0, x_2 = 0 \quad \rightarrow \quad O$

3. The optimal solution: $x_1^* = 1, \quad x_2^* = 1, \quad x_3^* = 0, \quad z^* = 7.$

4. $z_4 - c_4 = 1$, $z_5 - c_5 = 2$ and $z_6 - c_6 = 1$ are positive, and thus, it is proved that the basic solution that corresponds to the basis $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3)$ is optimal. The optimal solution: $x_1^* = 1, \quad x_2^* = 3, \quad x_3^* = 1, \quad z^* = 13.$

5. Verify the correctness of the two columns and compute the missing values.

   5.1 Vector $\mathbf{y}_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ is not correct, because $\mathbf{y}_1 = \mathbf{B}^{-1}\mathbf{a}_1$ does not hold.

$$\mathbf{y}_1 = \mathbf{B}^{-1}\mathbf{a}_1 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

5.2 Vector $\mathbf{y}_5 = \begin{pmatrix} -2 \\ \frac{1}{2} \\ -1 \end{pmatrix}$ is correct, because $\mathbf{y}_5 = \mathbf{B}^{-1}\mathbf{a}_5$ holds.

$$\mathbf{y}_5 = \mathbf{B}^{-1}\mathbf{a}_5 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ \frac{1}{2} \\ -1 \end{pmatrix}$$

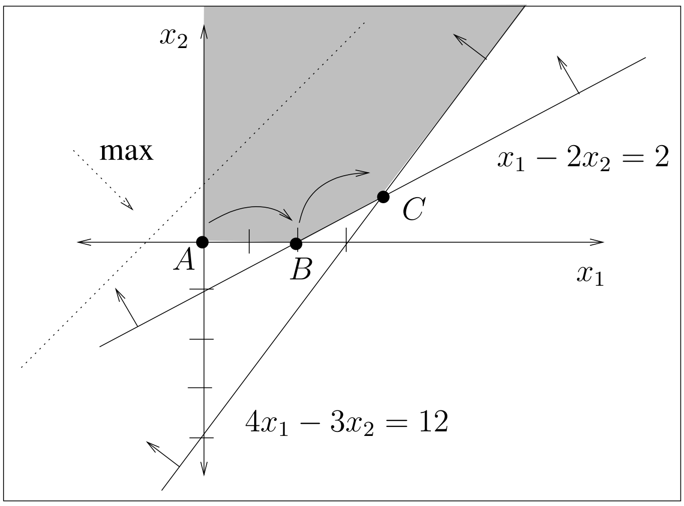
5.3 The missing values:

$$\mathbf{y}_3 = \begin{pmatrix} 4 \\ \frac{1}{2} \\ 2 \end{pmatrix}, \quad z_3 - c_3 = -1, \quad \mathbf{x}_B = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \quad z = 8.$$

6. The solution to the linear models and the identification of the extreme points in the graphical representation.

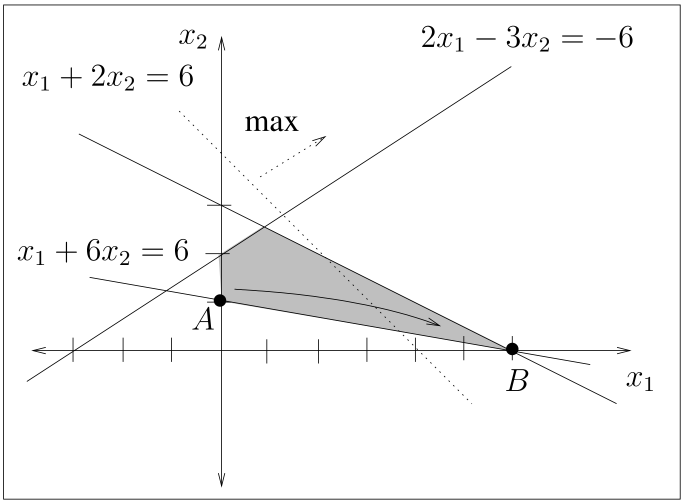
6.1 The optimal solution is unique, $x_1^* = \frac{18}{5}, x_2^* = \frac{4}{5}$.

- $\mathbf{B} = (\mathbf{a}_3\ \mathbf{a}_4), \quad \mathbf{x}_B = \begin{pmatrix} 2 \\ 12 \end{pmatrix}, \quad x_1 = 0, x_2 = 0 \quad \rightarrow \quad A$
- $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_4), \quad \mathbf{x}_B = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \quad x_1 = 2, x_2 = 0 \quad \rightarrow \quad B$
- $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_2), \quad \mathbf{x}_B = \begin{pmatrix} \frac{18}{5} \\ \frac{4}{5} \end{pmatrix}, \quad x_1^* = \frac{18}{5}, x_2^* = \frac{4}{5} \quad \rightarrow \quad C$



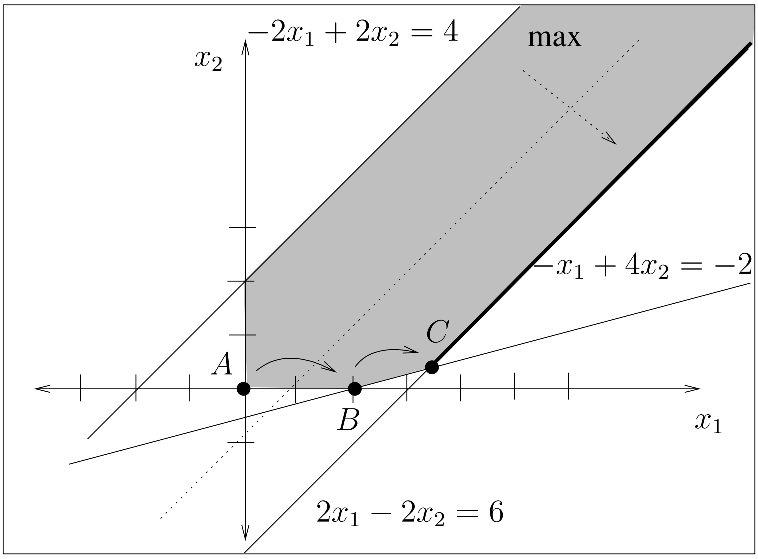
6.2 The optimal solution is unique and degenerate, $x_1^* = 6, x_2^* = 0, z^* = 6$.

- $\mathbf{B} = (\mathbf{a}_{w1}\ \mathbf{a}_4\ \mathbf{a}_5), \quad \mathbf{x}_B = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$
- $\mathbf{B} = (\mathbf{a}_2\ \mathbf{a}_4\ \mathbf{a}_5), \quad \mathbf{x}_B = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \quad x_1 = 0, x_2 = 1 \quad \rightarrow \quad A$
- $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_4\ \mathbf{a}_5), \quad \mathbf{x}_B = \begin{pmatrix} 6 \\ 18 \\ 0 \end{pmatrix}, \quad x_1 = 6, x_2 = 0 \quad \rightarrow \quad B$
- $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_4\ \mathbf{a}_3), \quad \mathbf{x}_B = \begin{pmatrix} 6 \\ 18 \\ 0 \end{pmatrix}, \quad x_1^* = 6, x_2^* = 0 \quad \rightarrow \quad B$



6.3 There are multiple optimal solutions, $z^* = 12$.

- $\mathbf{B} = (\mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5), \quad \mathbf{x}_B = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}, \quad x_1 = 0, x_2 = 0 \quad \rightarrow \quad A$
- $\mathbf{B} = (\mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_1), \quad \mathbf{x}_B = \begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix}, \quad x_1 = 2, x_2 = 0 \quad \rightarrow \quad B$
- $\mathbf{B} = (\mathbf{a}_3\ \mathbf{a}_2\ \mathbf{a}_1), \quad \mathbf{x}_B = \begin{pmatrix} 10 \\ \frac{1}{3} \\ \frac{10}{3} \end{pmatrix}, \quad x_1^* = \frac{10}{3}, x_2^* = \frac{1}{3} \quad \rightarrow \quad C$



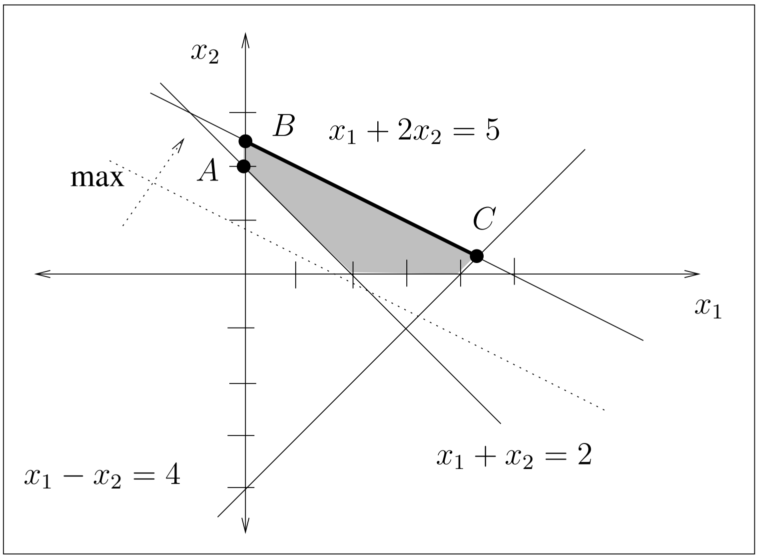
6.4 There are multiple optimal solutions, $z^* = 5$.

- $\mathbf{B} = (\mathbf{a}_3\ \mathbf{a}_{w1}\ \mathbf{a}_5), \quad \mathbf{x}_B = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix}$

- $\mathbf{B} = (\mathbf{a}_3\ \mathbf{a}_2\ \mathbf{a}_5), \quad \mathbf{x}_B = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \quad x_1 = 0, x_2 = 2 \quad \rightarrow \quad A$

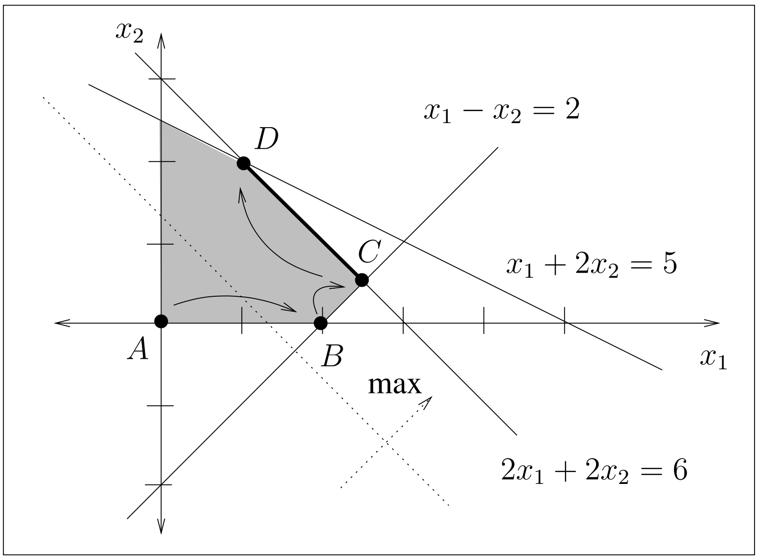
- $\mathbf{B} = (\mathbf{a}_4\ \mathbf{a}_2\ \mathbf{a}_5), \quad \mathbf{x}_B = \begin{pmatrix} \frac{1}{2} \\ \frac{5}{2} \\ \frac{13}{2} \end{pmatrix}, \quad x_1^* = 0, x_2^* = \frac{5}{2} \quad \rightarrow \quad B$

- $\mathbf{B} = (\mathbf{a}_4\ \mathbf{a}_2\ \mathbf{a}_1), \quad \mathbf{x}_B = \begin{pmatrix} \frac{8}{3} \\ \frac{1}{3} \\ \frac{13}{3} \end{pmatrix}, \quad x_1^* = \frac{13}{3}, x_2^* = \frac{1}{3} \quad \rightarrow \quad C$



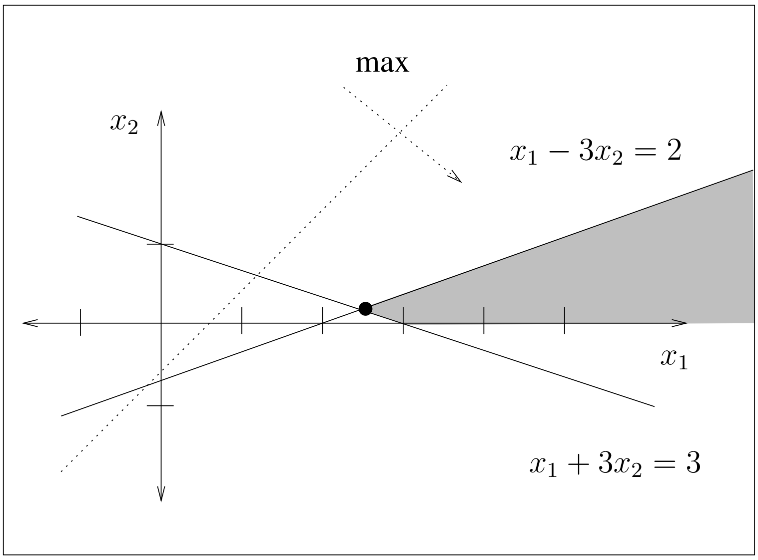
6.5 There are multiple optimal solutions, $z^* = 6$.

- $\mathbf{B} = (\mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5)$, $\quad \mathbf{x}_B = \begin{pmatrix} 2 \\ 6 \\ 5 \end{pmatrix}$, $\quad x_1 = 0, x_2 = 0 \quad \rightarrow \quad A$
- $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_4\ \mathbf{a}_5)$, $\quad \mathbf{x}_B = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$, $\quad x_1 = 2, x_2 = 0 \quad \rightarrow \quad B$
- $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_5)$, $\quad \mathbf{x}_B = \begin{pmatrix} \frac{5}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}$, $\quad x_1^* = \frac{5}{2}, x_2^* = \frac{1}{2} \quad \rightarrow \quad C$
- $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3)$, $\quad \mathbf{x}_B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\quad x_1^* = 1, x_2^* = 2 \quad \rightarrow \quad D$



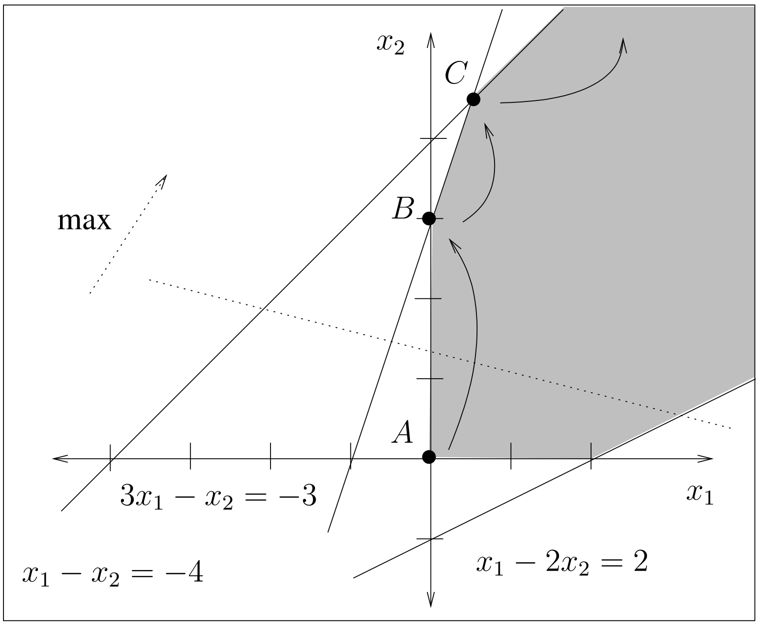
6.6 The solution is unbounded.

- $\mathbf{B} = (\mathbf{a}_{w1}\ \mathbf{a}_{w2}), \quad \mathbf{x}_B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$
- $\mathbf{B} = (\mathbf{a}_{w1}\ \mathbf{a}_{1}), \quad \mathbf{x}_B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- $\mathbf{B} = (\mathbf{a}_{2}\ \mathbf{a}_{1}), \quad \mathbf{x}_B = \begin{pmatrix} \frac{1}{6} \\ \frac{5}{2} \end{pmatrix}, \quad x_1 = \frac{5}{2}, x_2 = \frac{1}{6}$



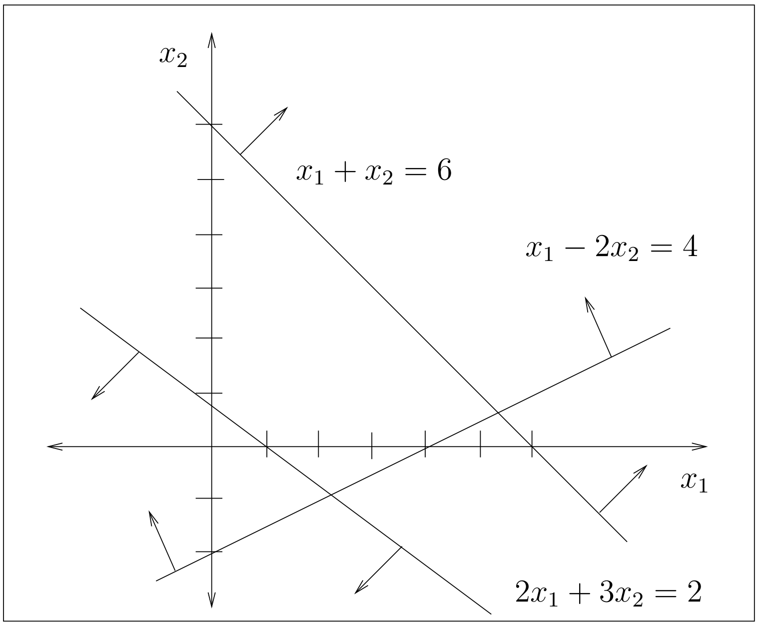
6.7 The solution is unbounded.

- $\mathbf{B} = (\mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5), \quad \mathbf{x}_B = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \quad x_1 = 0, x_2 = 0 \quad \rightarrow \quad A$
- $\mathbf{B} = (\mathbf{a}_3\ \mathbf{a}_2\ \mathbf{a}_5), \quad \mathbf{x}_B = \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix}, \quad x_1 = 0, x_2 = 3 \quad \rightarrow \quad B$
- $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_5), \quad \mathbf{x}_B = \begin{pmatrix} \frac{1}{2} \\ \frac{9}{2} \\ \frac{21}{2} \end{pmatrix}, \quad x_1 = \frac{1}{2}, x_2 = \frac{9}{2} \quad \rightarrow \quad C$



6.8 The problem is infeasible.

- $\mathbf{B} = (\mathbf{a}_3\ \mathbf{a}_{w1}\ \mathbf{a}_5), \quad \mathbf{x}_B = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}$
- $\mathbf{B} = (\mathbf{a}_3\ \mathbf{a}_{w1}\ \mathbf{a}_2), \quad \mathbf{x}_B = \begin{pmatrix} \frac{16}{3} \\ \frac{16}{3} \\ \frac{2}{3} \end{pmatrix}$
- $\mathbf{B} = (\mathbf{a}_3\ \mathbf{a}_{w1}\ \mathbf{a}_1), \quad \mathbf{x}_B = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$



7. The solution to the linear models.

   7.1 A unique optimal solution, $x_1^* = 0, \quad x_2^* = 8, \quad x_3^* = 0, \quad z^* = 16.$

   7.2 A unique optimal solution, $x_1^* = -1, \quad x_2^* = 7, \quad x_3^* = 0, \quad z^* = -12.$

   7.3 A unique optimal solution, $x_1^* = 5, \quad x_2^* = 6, \quad x_3^* = 0, \quad z^* = -13.$

   7.4 Multiple optimal solutions, $z^* = 24.$

   7.5 Multiple optimal solutions, $z^* = -73.$

   7.6 The problem is infeasible.

   7.7 Unbounded solution.

   7.8 The problem is infeasible.