Integer Programming. Exercises

1. Solve the following IP problems using the graphical solution:

1.1
$$\begin{aligned} \max\ z &= x_1 + 4x_2 \\ \text{subject to} & \\ x_1 + x_2 &\leq 7 \\ -x_1 + 3x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \text{ and integer} \end{aligned}$$

1.2
$$\begin{aligned} \max\ z &= 6x_1 + 8x_2 \\ \text{subject to} & \\ 2x_1 + 4x_2 &\leq 36 \\ 3x_1 - 4x_2 &\leq 40 \\ x_1, x_2 &\geq 0 \text{ and integer} \end{aligned}$$

2. Consider the following IP problems and the optimal tableaux for the LP relaxations associated to the IP problems:

2.1 Solve the following IP problem using the branch and bound algorithm. Choose x_1 as branching variable in the first branching.

$$\begin{aligned} \max\ z &= x_1 + 4x_2 \\ \text{subject to} & \\ x_1 + x_2 &\leq 7 \\ -x_1 + 3x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \text{ and integer} \end{aligned}$$

	x_1	x_2	x_3	x_4	
	0	0	$\frac{7}{4}$	$\frac{3}{4}$	$\frac{29}{2}$
$\mathbf{a}_1$	1	0	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{9}{2}$
$\mathbf{a}_2$	0	1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{2}$

2.2 Solve the following IP problem using the branch and bound algorithm. Choose x_1 as branching variable in the first branching.

$$\begin{aligned} \max\ z &= 6x_1 + 8x_2 \\ \text{subject to} & \\ 2x_1 + 4x_2 &\leq 36 \\ 3x_1 - 4x_2 &\leq 40 \\ x_1, x_2 &\geq 0 \text{ and integer} \end{aligned}$$

	x_1	x_2	x_3	x_4	
	0	0	$\frac{12}{5}$	$\frac{2}{5}$	$\frac{512}{5}$
$\mathbf{a}_2$	0	1	$\frac{3}{20}$	$-\frac{1}{10}$	$\frac{7}{5}$
$\mathbf{a}_1$	1	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{76}{5}$

2.3 Solve the following IP problem using the branch and bound algorithm.

$$\begin{aligned} \max\ z &= 2x_1 + x_2 + 3x_3 \\ \text{subject to} & \\ x_1 + x_2 + 3x_3 &\leq 17 \\ 3x_1 + 2x_2 + 2x_3 &\leq 11 \\ x_1, x_2, x_3 &\geq 0 \text{ and integer} \end{aligned}$$

	x_1	x_2	x_3	x_4	x_5	
	$\frac{5}{2}$	2	0	0	$\frac{3}{2}$	$\frac{33}{2}$
$\mathbf{a}_4$	$-\frac{7}{2}$	-2	0	1	$-\frac{3}{2}$	$\frac{1}{2}$
$\mathbf{a}_3$	$\frac{3}{2}$	1	1	0	$\frac{1}{2}$	$\frac{11}{2}$

2.4 Solve the following IP problem using the branch and bound algorithm. If there are several choices, always choose the variable with the smallest subindex as branching variable.

$$\max\ z = x_1 + x_2 + x_3$$

subject to

$$5x_1 + 2x_2 + 3x_3 \leq 42$$
$$2x_1 + 7x_2 + 5x_3 \leq 52$$
$$x_1, x_2, x_3 \geq 0 \text{ and integer}$$

	x_1	x_2	x_3	x_4	x_5	
	0	$\frac{1}{19}$	0	$\frac{3}{19}$	$\frac{2}{19}$	$\frac{230}{19}$
$\mathbf{a}_1$	1	$-\frac{11}{19}$	0	$\frac{5}{19}$	$-\frac{3}{19}$	$\frac{54}{19}$
$\mathbf{a}_3$	0	$\frac{31}{19}$	1	$-\frac{2}{19}$	$\frac{5}{19}$	$\frac{176}{19}$

3. Solve the following 0-1 IP problems using the 0-1 branch and bound algorithm.

3.1 $\max\ z = 8x_1 + 2x_2 + 3x_3 + x_4 + 6x_5$

subject to

$$2x_1 + x_2 + 4x_3 + x_4 + 4x_5 \leq 10$$
$$3x_1 + 2x_2 + 2x_3 + x_4 + 5x_5 \leq 11$$
$$x_1, x_2, x_3, x_4, x_5 = 0 \text{ or } 1$$

3.2 $\max\ z = 9x_1 + x_2 + 5x_3 + 2x_4 + 4x_5$

subject to

$$2x_1 + 8x_2 + x_3 + x_4 + 2x_5 \leq 7$$
$$3x_1 + 5x_2 + 3x_3 + 2x_4 + x_5 \leq 6$$
$$x_1, x_2, x_3, x_4, x_5 = 0 \text{ or } 1$$

3.3 $\max\ z = 10x_1 + 2x_2 + 7x_3 + 6x_4 + 3x_5$

subject to

$$8x_1 + x_2 + 5x_3 + 4x_4 + 2x_5 \leq 14$$
$$6x_1 + x_2 + 3x_3 + 6x_4 + x_5 \leq 11$$
$$x_1, x_2, x_3, x_4, x_5 = 0 \text{ or } 1$$

3.4 $\max\ z = -x_1 + 4x_2 - 2x_3 + 3x_4 + 7x_5 + 6x_6$

subject to

$$x_1 + 4x_2 + 3x_3 + 2x_4 + 2x_5 + 4x_6 \leq 11$$
$$2x_1 + 5x_2 + 2x_3 + 6x_4 + 8x_5 + 4x_6 \leq 19$$
$$x_1, x_2, x_3, x_4, x_5, x_6 = 0 \text{ or } 1$$

3.5 $\max\ z = 2x_1 - x_2 - 3x_3 + 5x_4$

$$\begin{aligned} &\text{subject to} \\ &5x_1 + x_2 + 2x_3 + 7x_4 \leq 10 \\ &x_1 + x_2 + x_3 + x_4 \geq 3 \\ &x_1, x_2, x_3, x_4 = 0 \text{ or } 1 \end{aligned}$$

4. Six components, $C_1, C_2, C_3, C_4, C_5, C_6$, must be carried in a box that can hold up to 15 kg. The value and the weight associated to each of the components are listed below:

	C_1	C_2	C_3	C_4	C_5	C_6
Value (euro)	4	2	1	7	3	6
Weight (kg)	5	8	8	6	1	5

At least 3 components must be carried in the box. Since the objective is to maximize the total value of the components introduced in the box and the weight capacity does not allow to carry them all, we need to choose some of them. The following binary variables have been defined:

$$x_i = \begin{cases} 1 & \text{if component } C_i \text{ is selected to be carried in the box} \\ 0 & \text{otherwise} \end{cases}$$

The 0-1 IP model that represents the problem is:

$$\begin{aligned} \max\ z = 4x_1 + 2x_2 + x_3 + 7x_4 + 3x_5 + 6x_6 \\ \text{subject to} \\ 5x_1 + 8x_2 + 8x_3 + 6x_4 + x_5 + 5x_6 \leq 15 \\ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3 \\ x_1, x_2, x_3, x_4, x_5, x_6 = 0 \text{ or } 1 \end{aligned}$$

Solve the problem using the 0-1 branch and bound algorithm, and determine which of the 6 components will be selected to be carried in the box so as to maximize the total value of the selected components.