Sensitivity analysis. Exercises

1. Consider the following linear model and its corresponding optimal tableau:

$$\max\ z = 4x_1 + x_2 + 5x_3$$

subject to

$$x_1 + x_2 + x_3 \le 4$$
$$2x_1 + x_2 + 3x_3 \le 10$$
$$3x_1 + x_2 + 4x_3 \le 16$$
$$x_1, x_2, x_3 \ge 0$$

	x_1	x_2	x_3	x_4	x_5	x_6	
	0	2	0	2	1	0	18
$\mathbf{a}_1$	1	2	0	3	−1	0	2
$\mathbf{a}_3$	0	−1	1	−2	1	0	2
$\mathbf{a}_6$	0	−1	0	−1	−1	1	2

1.1 Analyze how the following discrete changes affect the optimal tableau. For each of the changes, determine the optimal solution to the new linear model.

1.1.1 $\mathbf{b} = \begin{pmatrix} 4 \\ 10 \\ 16 \end{pmatrix} \quad \to \quad \hat{\mathbf{b}} = \begin{pmatrix} 2 \\ 10 \\ 16 \end{pmatrix}$

1.1.2 $\mathbf{b} = \begin{pmatrix} 4 \\ 10 \\ 16 \end{pmatrix} \quad \to \quad \hat{\mathbf{b}} = \begin{pmatrix} 4 \\ 10 \\ 18 \end{pmatrix}$

1.1.3 $\mathbf{c}^T = (4\ \ 1\ \ 5) \quad \to \quad \hat{\mathbf{c}}^T = (3\ \ 3\ \ 5)$

1.1.4 $\mathbf{c}^T = (4\ \ 1\ \ 5) \quad \to \quad \hat{\mathbf{c}}^T = (5\ \ 1\ \ 7)$

1.1.5 $\mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \to \quad \hat{\mathbf{a}}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

1.1.6 $\mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \to \quad \hat{\mathbf{a}}_2 = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$

1.1.7 New variable: x_4 $\quad c_4 = 6 \quad \mathbf{a}_4 = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$

1.1.8 New variable: x_4 $\quad c_4 = 3 \quad \mathbf{a}_4 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$

1.1.9 New constraint: $2x_1 + 4x_2 + x_3 \leq 8$

1.1.10 New constraint: $4x_1 + x_2 + 2x_3 \leq 8$

1.2 Find and interpret the shadow prices for the three resources.

1.3 For each of the components in vectors $\mathbf{c}$ and $\mathbf{b}$, find the range of values that leaves the current basis unchanged.

2. Consider the following linear model and its corresponding optimal tableau:

$$
\begin{aligned}
\max\ z = 4x_1 + 6x_2 + 5x_3 & \\
\text{subject to} & \\
x_1 + 2x_2 + 2x_3 &\leq 12 \\
2x_1 + 4x_2 + 2x_3 &\leq 14 \\
x_1 + x_2 + 2x_3 &\leq 6 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

	x_1	x_2	x_3	x_4	x_5	x_6	
	0	0	1	0	1	2	26
$\mathbf{a}_4$	0	0	1	1	$-\frac{1}{2}$	0	5
$\mathbf{a}_2$	0	1	-1	0	$\frac{1}{2}$	-1	1
$\mathbf{a}_1$	1	0	3	0	$-\frac{1}{2}$	2	5

2.1 Analyze how the following discrete changes affect the optimal tableau. For each of the changes, determine the optimal solution to the new linear model.

2.1.1 $\mathbf{b} = \begin{pmatrix} 12 \\ 14 \\ 6 \end{pmatrix} \rightarrow \hat{\mathbf{b}} = \begin{pmatrix} 7 \\ 14 \\ 6 \end{pmatrix}$

2.1.2 $\mathbf{b} = \begin{pmatrix} 12 \\ 14 \\ 6 \end{pmatrix} \rightarrow \hat{\mathbf{b}} = \begin{pmatrix} 12 \\ 18 \\ 10 \end{pmatrix}$

2.1.3 $\mathbf{c}^T = (4\ \ 6\ \ 5) \rightarrow \hat{\mathbf{c}}^T = (6\ \ 8\ \ 2)$

2.1.4 $\mathbf{c}^T = (4\ \ 6\ \ 5) \rightarrow \hat{\mathbf{c}}^T = (4\ \ 6\ \ 9)$

2.1.5 $\mathbf{a}_3 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \rightarrow \hat{\mathbf{a}}_3 = \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$

2.1.6 $\mathbf{a}_3 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \quad \rightarrow \quad \hat{\mathbf{a}}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

2.1.7 New variable: x_4 $\quad c_4 = 2 \quad \mathbf{a}_4 = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}$

2.1.8 New variable: x_4 $\quad c_4 = 5 \quad \mathbf{a}_4 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

2.1.9 New constraint: $x_1 + 2x_2 \leq 6$

2.1.10 New constraint: $x_1 + 3x_2 + 2x_3 \leq 10$

2.2 Find and interpret the shadow prices for the three resources.

2.3 For each of the components in vectors **c** and **b**, find the range of values that leaves the current basis unchanged.

3. A publisher wants to design three types of cookery books: B_1, B_2 and B_3. Cooks specialized in different areas have been contracted: 40 are cooks specialized in daily menus, 20 are pastry cooks and 10 are expert appetizer cooks. Cookbooks of type B_1 contain all kind of recipes. However, cookbooks of type B_2 do not contain appetizer recipes, and cookbooks of type B_3 do not contain pastry recipes. A team of 5 cooks is needed to produce a new design of cookery book. The following table shows the number of cook experts involved in each of the teams.

Type of cookery book	Expert cooks		
	Daily menu cooks	Pastry cooks	Appetizer cooks
B_1	2	2	1
B_2	4	1	0
B_3	4	0	1
Total number of cooks	40	20	10

Each of the 5 cook teams will design one cookery book. Considering the total amount of cooks available, the publisher needs to determine the number of teams that can be formed, and therefore, the number of type B_i cookery books that will be designed.

The following decision variables have been defined:

$$x_i : \text{number of type } B_i \text{ cookery books designed}, (i = 1, 2, 3).$$

The publisher obtains the same benefit from the three types of cookery books designed. The aim is to maximize the benefit. The following linear model has been written to represent the problem, and after adding three slack variables to the constraints and solving it, the optimal tableau shown below has been obtained.

$$\begin{array}{ll} \max & z = x_1 + x_2 + x_3 \\ \text{subject to} & \\ & 2x_1 + 4x_2 + 4x_3 \leq 40 \\ & 2x_1 + x_2 \leq 20 \\ & x_1 + x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

	x_1	x_2	x_3	x_4	x_5	x_6	
	0	0	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	14
$\mathbf{a}_2$	0	1	0	$\frac{1}{5}$	$\frac{1}{5}$	$-\frac{4}{5}$	4
$\mathbf{a}_1$	1	0	0	$-\frac{1}{10}$	$\frac{2}{5}$	$\frac{2}{5}$	8
$\mathbf{a}_3$	0	0	1	$\frac{1}{10}$	$-\frac{2}{5}$	$\frac{3}{5}$	2

3.1 Look at the model and its corresponding optimal tableau, and determine the number of teams that will work on designing cookery books of each type, that is to say, the number of B_i type cookery books that will be designed. What is the optimal objective value obtained?

3.2 Do all the cooks contracted take part in a team?

3.3 If the benefit obtained from each type B_1 cookery book is twice the benefit obtained from type B_2 or type B_3 ones, $c_1 = 2$, will the number of B_i type cookery books designed change? And, if it is three times higher, $c_1 = 3$? How much can c_1 be increased so that the number of type B_i cookery books designed will remain unchanged?

3.4 If the number of pastry cooks are 30 instead of 20, the teams will be affected by the change. How many B_i type cookery books will be designed? What will the optimal objective value be in that case?

4. An enterprise produces four fashion colors, C_1, C_2, C_3 and C_4, by mixing together three basic colors: red, blue and yellow. There are 26 kg red paint, 14 kg blue paint and 32 kg yellow paint available. The new colors are obtained by mixing the basic ones as follows:

$$\begin{array}{lcl} \text{To produce 1 kg } C_1 & \rightarrow & \frac{1}{2} \text{ kg red} + \frac{1}{4} \text{ kg blue} + \frac{1}{4} \text{ kg yellow} \\ \text{To produce 1 kg } C_2 & \rightarrow & \frac{3}{8} \text{ kg red} + \frac{1}{4} \text{ kg blue} + \frac{3}{8} \text{ kg yellow} \\ \text{To produce 1 kg } C_3 & \rightarrow & \frac{1}{3} \text{ kg red} + \frac{1}{3} \text{ kg blue} + \frac{1}{3} \text{ kg yellow} \\ \text{To produce 1 kg } C_4 & \rightarrow & \frac{3}{10} \text{ kg red} + \frac{2}{5} \text{ kg blue} + \frac{3}{10} \text{ kg yellow} \end{array}$$

The sales department manager knows that a benefit of 3, 4, 1 and 6 is obtained from 1 kg C_1, C_2, C_3 and C_4 paint, respectively. Since the aim is to maximize benefit, s/he wants to know the amount of C_1, C_2, C_3 and C_4 paint that should be produced. The linear model shown below has been formulated to represent the problem, and its corresponding optimal tableau has been found.

$$\begin{array}{rl} \max & z = 3x_1 + 4x_2 + x_3 + 6x_4 \\ \text{subject to} & \\ & \frac{1}{2}x_1 + \frac{3}{8}x_2 + \frac{1}{3}x_3 + \frac{3}{10}x_4 \leq 26 \\ & \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{3}x_3 + \frac{2}{5}x_4 \leq 14 \\ & \frac{1}{4}x_1 + \frac{3}{8}x_2 + \frac{1}{3}x_3 + \frac{3}{10}x_4 \leq 32 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
	1	0	$\frac{13}{3}$	$\frac{2}{5}$	0	16	0	224
$\mathbf{a}_5$	$\frac{1}{8}$	0	$-\frac{1}{6}$	$-\frac{3}{10}$	1	$-\frac{3}{2}$	0	5
$\mathbf{a}_2$	1	1	$\frac{4}{3}$	$\frac{8}{5}$	0	4	0	56
$\mathbf{a}_7$	$-\frac{1}{8}$	0	$-\frac{1}{6}$	$-\frac{3}{10}$	0	$-\frac{3}{2}$	1	11

4.1 Is the enterprise buying the appropriate amount of basic paint (red, blue, yellow)? How could these quantities be modified to increase the benefit? Find and interpret the shadow prices for the three resources.

4.2 The sales department manager suspects that there is not enough blue paint. Find the maximum amount of blue paint for which the current basis remains unchanged, and therefore, the tableau remains optimal.

4.3 Let us suppose that s/he decides to buy more blue paint and less red and yellow paint. The resources available will change as follows:

$$\mathbf{b} = \begin{pmatrix} 26 \\ 14 \\ 32 \end{pmatrix} \quad \rightarrow \quad \hat{\mathbf{b}} = \begin{pmatrix} 24 \\ 20 \\ 31 \end{pmatrix}$$

What is the optimal amount of C_1, C_2, C_3 and C_4 paint to be produced to maximize the benefit?