The Simplex Method. Exercises

1. Write the following linear models in maximization standard form:

1.1
$$\begin{aligned} \max\ z = 2x_1 + 4x_2 - 4x_3 \\ \text{subject to} \\ 3x_1 + 2x_2 + 4x_3 \geq 1 \\ 4x_1 - 3x_2 = 2 \\ 2x_1 + x_2 + 6x_3 \leq 3 \\ x_1, x_2 \geq 0, x_3 : \text{unrestricted} \end{aligned}$$

1.2
$$\begin{aligned} \min\ z = 2x_1 - 3x_2 + x_3 \\ \text{subject to} \\ x_1 - 5x_2 + 6x_3 \geq 8 \\ x_1 - 4x_2 \leq -12 \\ 2x_1 - x_2 + 4x_3 = 5 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

1.3
$$\begin{aligned} \min\ z = 2x_1 + 2x_2 - 4x_3 \\ \text{subject to} \\ 2x_1 + 2x_2 + 2x_3 = 10 \\ -2x_1 + 6x_2 - x_3 \leq -10 \\ -x_1 + 3x_2 \geq 3 \\ x_1 \leq 0, x_2, x_3 \geq 0 \end{aligned}$$

1.4
$$\begin{aligned} \max\ z = 3x_1 - 7x_2 + 5x_3 \\ \text{subject to} \\ x_2 - x_3 \leq -9 \\ -x_1 - 2x_3 \geq 5 \\ 4x_1 - x_2 = 6 \\ x_1 \leq 0, x_2 \geq 0, x_3 : \text{unrestricted} \end{aligned}$$

2. Consider the following linear model:

$$\begin{aligned} \max\ z = x_1 + x_2 \\ \text{subject to} \\ -x_1 + x_2 \leq 4 \\ 2x_1 + 5x_2 \leq 20 \\ 2x_1 - x_2 \leq 2 \\ x_1, x_2 \geq 0 \end{aligned}$$

2.1 Solve the model using the graphical solution.

2.2 Compute all the basic solutions. Which of them are feasible? Which of them are degenerate?

2.3 Associate each of the basic solutions computed in 2.2 to their corresponding point in the graphical representation.

3. Consider the following linear model:

$$\begin{aligned} \max\ z = 4x_1 + 3x_2 + 2x_3 \\ \text{subject to} \\ x_1 + 2x_2 + 3x_3 \leq 6 \\ 2x_1 + x_2 + x_3 \leq 3 \\ x_1 + x_2 + x_3 \leq 2 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

Write the model in maximization standard form, compute the basic feasible solution associated with the basis $\mathbf{B} = (\mathbf{a}_4\ \mathbf{a}_1\ \mathbf{a}_6)$, and apply the improvement theorem to find the optimal solution.

4. Consider the following linear model:

$$\begin{aligned} \max\ z = 2x_1 + 2x_2 + 5x_3 \\ \text{subject to} \\ x_1 \quad\quad + x_3 \leq 2 \\ x_2 + x_3 \leq 4 \\ x_1 \quad + 2x_3 \leq 3 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

Given that the inverse of the following matrix is known,

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

prove that the basic solution that corresponds to the basis $\mathbf{B} = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3)$ is optimal. Compute the optimal solution and the optimal objective value.

5. Consider the following linear model:

$$\begin{aligned} \max\ z = x_1 + 4x_2 + 3x_3 \\ \text{subject to} \\ 3x_1 + 4x_2 + 6x_3 \leq 10 \\ x_1 + 2x_2 + x_3 \leq 4 \\ 2x_1 + 2x_2 + 3x_3 \leq 8 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

Let us suppose that applying the simplex algorithm we obtain the following tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	
	1	0		0	2	0	
$\mathbf{a}_4$	2	0		1	-2	0	
$\mathbf{a}_2$	2	1		0	$\frac{1}{2}$	0	
$\mathbf{a}_6$	1	0		0	-1	1	

5.1 Prove that the column $\mathbf{y}_1$ has not been correctly computed.

5.2 Prove that the column $\mathbf{y}_5$ is correct.

5.3 Use the information in the model and in the tableau to compute the missing values.

6. Apply the simplex algorithm to solve the following linear models. If the model is feasible, show in the graphical representation the extreme points that correspond to the basic feasible solutions computed in the simplex tableaux.

6.1 $\max\ z = x_1 - x_2$

subject to

$$\begin{aligned} x_1 - 2x_2 &\leq 2 \\ 4x_1 - 3x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

6.2 $\max\ z = x_1 + x_2$

subject to

$$\begin{aligned} x_1 + 6x_2 &\geq 6 \\ 2x_1 - 3x_2 &\geq -6 \\ x_1 + 2x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

6.3 $\max\ z = 4x_1 - 4x_2$

subject to

$$\begin{aligned} -2x_1 + 2x_2 &\leq 4 \\ 2x_1 - 2x_2 &\leq 6 \\ -x_1 + 4x_2 &\geq -2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

6.4 $\max\ z = x_1 + 2x_2$

subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 5 \\ x_1 + x_2 &\geq 2 \\ x_1 - x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

6.5 $\max\ z = 2x_1 + 2x_2$

subject to

$$x_1 - x_2 \leq 2$$
$$2x_1 + 2x_2 \leq 6$$
$$x_1 + 2x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

6.6 $\max\ z = 2x_1 - 2x_2$

subject to

$$x_1 + 3x_2 \geq 3$$
$$x_1 - 3x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

6.7 $\max\ z = x_1 + 4x_2$

subject to

$$x_1 - x_2 \geq -4$$
$$3x_1 - x_2 \geq -3$$
$$x_1 - 2x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

6.8 $\max\ z = 3x_1 + 4x_2$

subject to

$$x_1 - 2x_2 \leq 4$$
$$x_1 + x_2 \geq 6$$
$$2x_1 + 3x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

7. Apply the simplex algorithm to solve the following linear models:

7.1 $\max\ z = 3x_1 + 2x_2 + x_3$

subject to

$$x_1 - x_2 + x_3 \leq 4$$
$$2x_1 + x_2 + 4x_3 \leq 8$$
$$x_1, x_2, x_3 \geq 0$$

7.2 $\min\ z = 5x_1 - x_2 - 2x_3$

subject to

$$x_1 + x_2 - x_3 = 6$$
$$x_1 + 2x_2 - x_3 \geq 4$$
$$-x_1 + x_2 + 2x_3 \leq 8$$
$$x_1 \leq 0, x_2, x_3 \geq 0$$

7.3 $\min\ z = x_1 - 3x_2 + 2x_3$

subject to

$$3x_1 - x_2 + 2x_3 \leq 9$$
$$-2x_1 + 4x_2 + x_3 \leq 14$$
$$-4x_1 + 4x_2 + 8x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

7.4 $\min\ z = 10x_1 + 8x_2 + 6x_3 + 4x_4$

subject to

$$2x_1 + 4x_2 + 2x_3 + x_4 \geq 10$$
$$-4x_1 + 4x_2 - x_3 + 2x_4 \geq 12$$
$$x_1, x_2, x_3, x_4 \geq 0$$

7.5 min $z = -16x_1 + 2x_2 + x_3 - 2x_4 - 3x_5$

subject to

$$3x_1 + x_2 + 3x_3 - 3x_4 + 9x_5 \leq 12$$

$$2x_1 + 8x_2 + 4x_3 + 2x_4 - 4x_5 \leq 10$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

7.6 max $z = 9x_1 + 5x_2 + 4x_3 + x_4$

subject to

$$2x_1 + x_2 + x_3 + 2x_4 \leq 2$$

$$8x_1 + 4x_2 - 2x_3 - x_4 \geq 10$$

$$4x_1 + 7x_2 + 2x_3 + x_4 \leq 4$$

$$x_1, x_2, x_3, x_4 \geq 0$$

7.7 min $z = -4x_1 + 2x_2 - x_3$

subject to

$$x_1 - x_2 + 2x_3 \geq -8$$

$$2x_1 - 3x_2 + 4x_3 \leq 5$$

$$x_1, x_2 \geq 0, x_3 : \text{unrestricted}$$

7.8 min $z = 3x_1 + x_2 - 2x_3 - 2x_4 + x_5$

subject to

$$x_1 + 2x_2 + 2x_3 + x_4 + x_5 \leq 2$$

$$2x_1 + x_2 + 3x_3 + 2x_4 + 2x_5 \geq 12$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$