A. LINEAR ALGEBRA. CONVEX SETS

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1. Matrices and vectors

The elements of $\mathbb{R}$ are called scalars.

An $m \times n$ matrix $\mathbf{A}$ is a rectangular array of scalars:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

If a matrix has just one column, $m \times 1$, it is said to be a column vector.

$$\mathbf{a} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$$

1.1 Matrix operations

Addition of matrices

Given matrices $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{m\times n}$, $\mathbf{B} = (b_{ij}) \in \mathbb{R}^{m\times n}$, the addition of $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A}+\mathbf{B}$, is the matrix $\mathbf{C} = (c_{ij}) \in \mathbb{R}^{m\times n}$ obtained by performing the addition componentwise:

$$c_{ij} = a_{ij} + b_{ij}, \quad i = 1, \ldots, m, \; j = 1, \ldots, n.$$

Properties

1. The addition of matrices is an inner operation.

$$\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n} \quad \Rightarrow \quad \mathbf{A}+\mathbf{B} \in \mathbb{R}^{m\times n}.$$

2. The addition of matrices is commutative.

$$\forall \mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n} \quad \mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}.$$

3. The addition of matrices is associative.

$$\forall \mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{m\times n} \quad (\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C}).$$

4. There exists a neutral element for the addition.

$$\forall \mathbf{A} \in \mathbb{R}^{m\times n} \quad \mathbf{A}+\mathbf{0} = \mathbf{0}+\mathbf{A} = \mathbf{A}, \text{ where } \mathbf{0} \in \mathbb{R}^{m\times n}.$$

5. There exist opposite elements for the addition.

$$\forall \mathbf{A} \in \mathbb{R}^{m\times n} \quad \mathbf{A}+(-\mathbf{A}) = (-\mathbf{A})+\mathbf{A} = \mathbf{0},$$

where $-\mathbf{A} \in \mathbb{R}^{m\times n}$.

Scalar multiplication

Let $\alpha \in \mathbb{R}$ be a scalar and $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{m\times n}$ a matrix. The operation of multiplying $\mathbf{A}$ by α, is represented by $\alpha \cdot \mathbf{A}$ and is performed by multiplying every element of $\mathbf{A}$ by α. The result is a matrix

$$\mathbf{B} = (b_{ij}) \in \mathbb{R}^{m\times n}$$

where

$$b_{ij} = \alpha \cdot a_{ij}, \quad i = 1, \ldots, m, \; j = 1, \ldots, n.$$

Inner product

The multiplication of a row vector $\mathbf{a}^T$ and a column vector $\mathbf{b}$ is called the inner product of the two vectors, and is denoted by $\mathbf{a}^T \cdot \mathbf{b}$. The result of this multiplication is a real number, and it is obtained in the following way:

$$\mathbf{a}^T = (a_1 \;\; \cdots \;\; a_n) \in \mathbb{R}^{1\times n}, \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n.$$

$$\mathbf{a}^T \cdot \mathbf{b} = (a_1 \;\; \cdots \;\; a_n) \cdot \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \sum_{i=1}^{n} a_i \cdot b_i.$$

Matrix multiplication

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ be an $m \times n$ matrix and $\mathbf{B} \in \mathbb{R}^{n\times p}$ an $n \times p$ matrix.

The product $\mathbf{A} \cdot \mathbf{B}$ is defined as the $m \times p$ matrix $\mathbf{C} = \mathbf{A} \cdot \mathbf{B} \in \mathbb{R}^{m\times p}$, where the entry (i, j) in $\mathbf{C}$ is the inner product of the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$.

Properties

1. It is associative. $\forall \mathbf{A} \in \mathbb{R}^{m\times n}, \forall \mathbf{B} \in \mathbb{R}^{n\times p}, \forall \mathbf{C} \in \mathbb{R}^{p\times q}$

$$(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}).$$

2. It is distributive with respect to the sum.

   $\forall \mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}, \ \forall \mathbf{C} \in \mathbb{R}^{n\times p}, \quad (\mathbf{A}+\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot\mathbf{C}+\mathbf{B}\cdot\mathbf{C}.$

   $\forall \mathbf{A} \in \mathbb{R}^{m\times n}, \ \forall \mathbf{B}, \mathbf{C} \in \mathbb{R}^{n\times p}, \quad \mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}.$

3. $\forall \mathbf{A} \in \mathbb{R}^{m\times n}, \quad \mathbf{A} \cdot \mathbf{0}_{n\times p} = \mathbf{0}_{m\times p}, \quad \mathbf{0}_{q\times m} \cdot \mathbf{A} = \mathbf{0}_{q\times n}.$

4. $\forall \mathbf{A} \in \mathbb{R}^{m\times n}, \quad \mathbf{I}_m \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{I}_n = \mathbf{A}.$

5. $\forall \alpha \in \mathbb{R}, \ \forall \mathbf{A} \in \mathbb{R}^{m\times n}, \ \forall \mathbf{B} \in \mathbb{R}^{n\times p},$

$$\alpha \cdot (\mathbf{A} \cdot \mathbf{B}) = (\alpha \cdot \mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (\alpha \cdot \mathbf{B}).$$

1.2 The rank of a matrix

Definition 1 *Let* $\mathbf{A} \in \mathbb{R}^{m \times n}$ *be a matrix, and let* $\mathbf{U}$ *be the resulting matrix after performing Gaussian elimination. The rank of matrix* $\mathbf{A}$ *is denoted by rank* $\mathbf{A}$ *and is equal to the number of pivot elements of matrix* $\mathbf{U}$*.*

2. Systems of linear equations

Let us consider a system with m linear equations of n variables

$$\mathbf{Ax} = \mathbf{b},$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, rank $\mathbf{A} = r$ and $\mathbf{b} \in \mathbb{R}^m$. We shall solve the system by performing Gaussian elimination.

The following cases may arise:

- rank $\mathbf{A} \neq$ rank $(\mathbf{A}\ \mathbf{b})$. The system has no solution. It is inconsistent.
- rank $\mathbf{A} =$ rank $(\mathbf{A}\ \mathbf{b}) = r$. There exists at least one solution to the system. It is consistent.
  - $r =$ number of variables. There exists a unique (one and only one) solution.
  - $r <$ number of variables. There exists an infinite number of solutions.

2.1. Basic solutions

Let $\mathbf{Ax} = \mathbf{b}$ be a system, where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $m < n$, and rank $\mathbf{A} =$ rank $(\mathbf{A}\ \mathbf{b}) = m$.

Assume that the first m columns of $\mathbf{A}$ are linearly independent. Let $\mathbf{B}$ be the $m \times m$ submatrix of $\mathbf{A}$ formed by its first m columns. Let $\mathbf{N}$ denote the last $n - m$ columns of $\mathbf{A}$. We can write the system $\mathbf{Ax} = \mathbf{b}$ in the following way:

$$(\mathbf{B} \quad \mathbf{N}) \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \mathbf{b},$$

or

$$\mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}.$$

Variables in $\mathbf{x}_B$: basic variables.

Variables in $\mathbf{x}_N$: nonbasic variables.

$$\mathbf{B}\mathbf{x}_B = \mathbf{b} - \mathbf{N}\mathbf{x}_N.$$

Setting all the nonbasic variables equal to zero, $\mathbf{x}_N = \mathbf{0}$, we obtain a system whose solution is unique:

$$\mathbf{B}\mathbf{x}_B = \mathbf{b}.$$

The solution thus calculated is called a basic solution of the system.

The maximum number of basic solutions:

$$\begin{pmatrix} n \\ m \end{pmatrix} = \frac{n!}{m!\ (n-m)!}$$

3. Vector spaces

Definition 2 (Linear combination) *A linear combination of vectors* $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n \in \mathbb{R}^m$ *is:*

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n,$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ *are real numbers.*

3.1 Linear dependence and independence

Definition 3 *The vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ *are said to be linearly independent if for every linear combination such that*

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$$

it implies that $\alpha_1 = \cdots = \alpha_n = 0$.

Definition 4 *The vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^m$ *are said to be linearly dependent if there exist* $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$ *not all of them zero, such that* $\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$. *Obviously, they are not linearly independent.*

3.2 Basis and dimension

Definition 5 *A set of vectors* $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \subseteq \mathbb{R}^m$ *is said to be a spanning set of* $\mathbb{R}^m$ *if every vector* $\mathbf{v} \in \mathbb{R}^m$ *can be represented as a linear combination of the vectors in* S*, that is, if there exist* $\alpha_1, \ldots, \alpha_p \in \mathbb{R}$ *such that*

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_p \mathbf{v}_p.$$

Definition 6 *A collection* $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subseteq \mathbb{R}^m$ *of vectors forms a basis in* $\mathbb{R}^m$ *if the following conditions hold:*

- *The vectors of* B *are linearly independent.*
- B *is a spanning set in* $\mathbb{R}^m$*.*

There are infinite bases in a vector space. However, all of them contain the same number of vectors. This number is the dimension of the vector space.

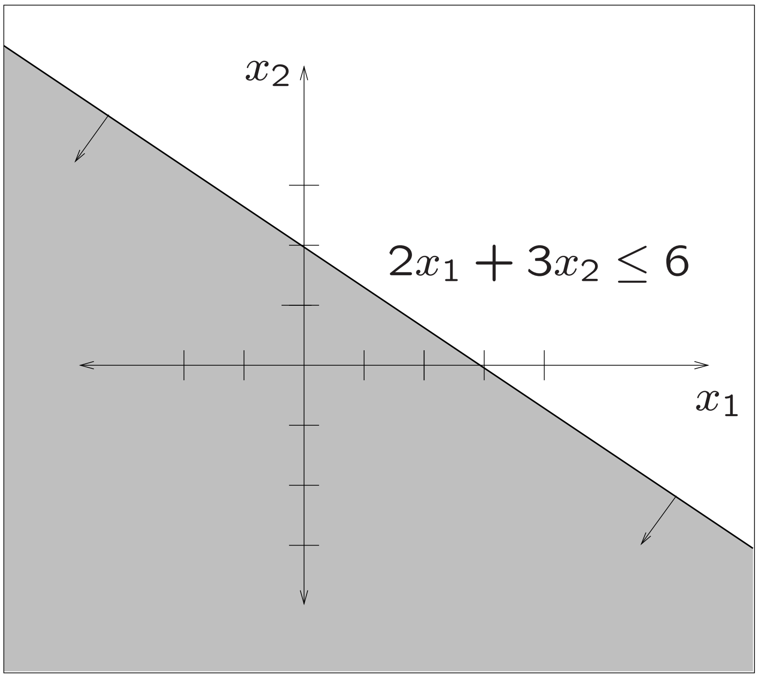
Theorem 1 *Let* $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ *be a basis in* $\mathbb{R}^m$*. Then, every vector* $\mathbf{v} \in \mathbb{R}^m$ *can be expressed as a linear combination of* $\mathbf{v}_1, \ldots, \mathbf{v}_m$*, and the coefficients of that linear combination are unique.*

Theorem 2 *Given a basis* B *for* $\mathbb{R}^m$ *and a vector* $\mathbf{v} \in \mathbb{R}^m$*,* $\mathbf{v} \notin B$ *and* $\mathbf{v} \neq \mathbf{0}$*, it is always possible to form another basis, by replacing a vector in* B *by the vector* $\mathbf{v}$*.*

4. Convex sets

An equation of the form $a_1x_1 + a_2x_2 = c$, where a_1, a_2 and c are constants, represents a straight line in $\mathbb{R}^2$.

An inequality of the form $a_1x_1 + a_2x_2 \leq c$ is the set of all points lying on the line $a_1x_1 + a_2x_2 = c$, together with all those points lying to one side of the line.



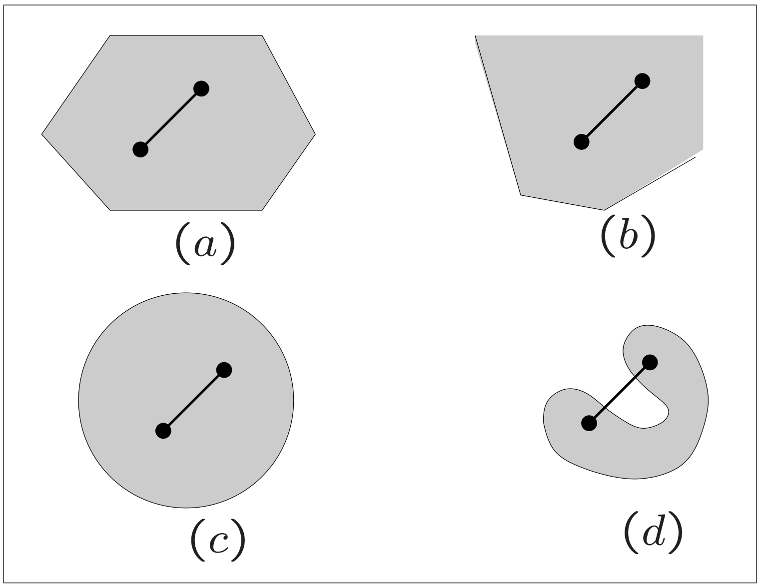
A half-space of $\mathbb{R}^2$ is the set of all points of $\mathbb{R}^2$ which satisfies an inequality of the form $a_1x_1 + a_2x_2 \leq c$ or $a_1x_1 + a_2x_2 \geq c$, where at least one of the constants a_1 or a_2 is nonzero.

In $\mathbb{R}^n$ the equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$, where a_1, ..., $a_n, c \in \mathbb{R}$ are constants, represents a hyperplane.

A half-space of $\mathbb{R}^n$ is the set of all points which satisfies an inequality of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq c$ or $a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq c$.

Definition 7 *A subset C of $\mathbb{R}^n$ is a convex set if C is empty, if C contains a single point, or if for every two distinct points in C, the line segment connecting them lies entirely in C.*

The sets (a), (b) and (c) in the Figure are convex, (d) is not convex.



The following results can be proved: (1) A hyperplane is a convex set. (2) A half-space is a convex set. (3) The intersection of a finite number of convex sets is a convex set.

In linear programming, convex sets such as hyperplanes, half-spaces and the intersection of a finite number of convex sets are of special significance, because they appear in the study of linear models.

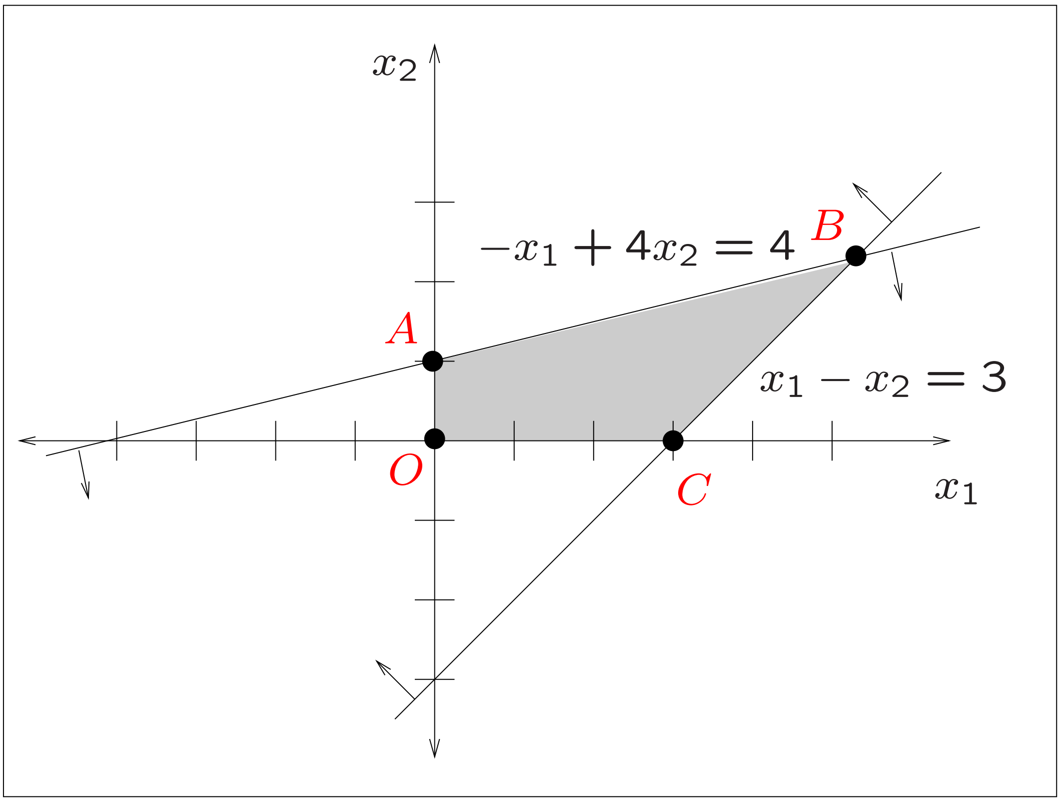
The intersection of a finite number of half-spaces is a convex set of the form (a), where the vertices of the set are called extreme points.

5. Extreme points and basic feasible solutions

Consider the following inequations, $x_1 \geq 0$ and $x_2 \geq 0$:

$$\begin{aligned} -x_1 + 4x_2 &\leq 4 \\ x_1 - x_2 &\leq 3 \end{aligned}$$

The intersection of the two half-spaces, together with $x_1 \geq 0$ and $x_2 \geq 0$, is a convex set; a polygon in this case. The polygon has a finite number of vertices: extreme points.



$O = (0, 0)$, $A = (0, 1)$, $B = (\frac{16}{3}, \frac{7}{3})$, $C = (3, 0)$.

Convert inequations into equations by adding nonnegative variables x_3 and x_4.

$$\begin{aligned} -x_1 + 4x_2 + x_3 \qquad &= 4 \\ x_1 - x_2 \qquad + x_4 &= 3 \end{aligned}$$

Compute the basic solutions and choose the ones with all the variables greater than or equal to zero; we can confirm that they correspond to the extreme points of the polygon.

- $x_3 = x_4 = 0$, solve $-x_1 + 4x_2 = 4$, $x_1 - x_2 = 3$: $x_1 = \frac{16}{3}$ and $x_2 = \frac{7}{3}$. Extreme point B.

- $x_2 = x_4 = 0$, solve $-x_1 + x_3 = 4$, $x_1 = 3$: $x_1 = 3$ and $x_3 = 7$. Extreme point C.

- $x_2 = x_3 = 0$, solve $-x_1 = 4$, $x_1 + x_4 = 3$: $x_1 = -4$ and $x_4 = 7$. It does not correspond to any extreme point.

- $x_1 = x_4 = 0$, solve $4x_2 + x_3 = 4$, $-x_2 = 3$: $x_2 = -3$ and $x_3 = 16$. It does not correspond to any extreme point.

- $x_1 = x_3 = 0$, solve $4x_2 = 4$, $-x_2 + x_4 = 3$: $x_2 = 1$ and $x_4 = 4$. Extreme point A.

- $x_1 = x_2 = 0$, solve $x_3 = 4$, $x_4 = 3$: $x_3 = 4$ and $x_4 = 3$. Extreme point O.