3. DUALITY

1. The dual problem

Definition 1 (Maximization symmetric form) *A linear problem is said to be written in maximization symmetric form if:*

- *The objective is in maximization form,*

- *All the constraints are $\leq$,*

- *All variables are nonnegative.*

Definition 2 (Minimization symmetric form) *A linear problem is said to be written in minimization symmetric form if:*

- *The objective is in minimization form,*

- *All the constraints are $\geq$,*

- *All variables are nonnegative.*

1.1 Primal-dual relationship

	Primal		Dual

$$\text{max } z = \mathbf{c}^T\mathbf{x} \qquad \text{min } G = \mathbf{b}^T\mathbf{y}$$

$$\text{subject to} \qquad\qquad \text{subject to}$$

$$\mathbf{A}\mathbf{x} \leq \mathbf{b} \qquad\qquad \mathbf{A}^T\mathbf{y} \geq \mathbf{c}$$

$$\mathbf{x} \geq \mathbf{0} \qquad\qquad\qquad \mathbf{y} \geq \mathbf{0}$$

1.2 Primal-dual correspondence

For any primal problem and the corresponding dual problem, there is a direct correspondence between the parameters of the problems:

- The primal constraint matrix $\mathbf{A}$ is $m \times n$; m constraints and n variables. The dual constraint matrix $\mathbf{A}^T$ is $n \times m$; n constraints and m variables.

- The right hand side vector $\mathbf{b}$ of the primal problem is the vector of cost coefficients of the dual.

- The vector of cost coefficients $\mathbf{c}$ of the primal is the right hand side vector of the dual.

- The number of constraints of the primal is equal to the number of variables of the dual.

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1.3 Duality. The general case

Objective function: max	$\Longleftrightarrow$	Objective function: min
ith constraint is $\leq b_i$	$\Longleftrightarrow$	ith variable is ≥ 0
ith constraint is $= b_i$	$\Longleftrightarrow$	ith var. is unrestricted
ith constraint is $\geq b_i$	$\Longleftrightarrow$	ith variable is ≤ 0
ith variable is ≥ 0	$\Longleftrightarrow$	ith constraint is $\geq b_i$
ith var. is unrestricted	$\Longleftrightarrow$	ith const. is $= b_i$
ith variable is ≤ 0	$\Longleftrightarrow$	ith constraint is $\leq b_i$

2. Duality proofs

All theorems are stated for the primal-dual symmetric forms.

Theorem 1 *The dual of the dual is the primal.*

Theorem 2 (Weak duality) *Let $\mathbf{x}$ and $\mathbf{y}$ be any feasible solutions to the primal and dual problems, respectively. The following inequality holds:*

$$z \;=\; \mathbf{c}^T\mathbf{x} \;\leq\; \mathbf{b}^T\mathbf{y} \;=\; G.$$

Corollary 1 *If $\mathbf{x}^*$ and $\mathbf{y}^*$ are feasible solutions to the primal and dual problem such that*

$$\mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*$$

holds, then $\mathbf{x}^*$ and $\mathbf{y}^*$ are optimal solutions to the primal and dual problems respectively.

Corollary 2 *If the primal problem is feasible and unbounded, the dual problem is infeasible.*

Corollary 3 *If the dual problem is feasible and unbounded, the primal problem is infeasible.*

If the primal problem is infeasible, the dual problem may be either infeasible or unbounded.

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Theorem 3 (The fundamental principle of duality) *If an optimal solution $\mathbf{x}^*$ to the primal problem exists, then an optimal solution $\mathbf{y}^*$ to the dual problem exists. Similarly, if an optimal solution $\mathbf{y}^*$ to the dual problem exists, then an optimal solution $\mathbf{x}^*$ to the primal problem exists. In both cases,*

$$z^* = \mathbf{c}^T\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^* = G^*$$

holds.

3. The principle of complementary slackness

We can use the primal optimal solution to solve the corresponding dual problem, and viceversa

Theorem 4 (Complementary slackness) *Any feasible solutions to the primal and dual problems* $\mathbf{x}^*$ *and* $\mathbf{y}^*$ *are optimal if and only if the following holds:*

$$\mathbf{x}^{*T}(\mathbf{A}^T\mathbf{y}^* - \mathbf{c}) + \mathbf{y}^{*T}(\mathbf{b} - \mathbf{A}\mathbf{x}^*) = 0.$$

3.1 Interpretation of the complementary slackness conditions

- $\mathbf{x}^* > \mathbf{0} \quad \Rightarrow \quad \mathbf{A}^T\mathbf{y}^* - \mathbf{c} = \mathbf{0}$.

- $\mathbf{A}\mathbf{x}^* < \mathbf{b} \quad \Rightarrow \quad \mathbf{y}^* = \mathbf{0}$.

- $\mathbf{y}^* > \mathbf{0} \quad \Rightarrow \quad \mathbf{A}\mathbf{x}^* - \mathbf{b} = \mathbf{0}$.

- $\mathbf{A}^T\mathbf{y}^* > \mathbf{c} \quad \Rightarrow \quad \mathbf{x}^* = \mathbf{0}$.

4. The optimal solution to the dual problem

Theorem 5 *Consider the symmetric form of duality, and let $\mathbf{B}$ be an optimal basis for the primal problem. Then,*

$$\mathbf{y}^{*T} = \mathbf{c}_B^T \mathbf{B}^{-1}$$

is an optimal solution to the dual problem.

4.1 A solution to the dual problem in the tableau

The reduced cost coefficients $z_j - c_j$ are computed as follows:

$$z_j - c_j = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{a}_j - c_j.$$

If we compute all the $z_j - c_j$ of the vectors $\mathbf{a}_j$ that correspond to the initial identity matrix $\mathbf{I}$, we get:

$$\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{I} - \mathbf{c}_I^T = \mathbf{c}_B^T \mathbf{B}^{-1} - \mathbf{c}_I^T.$$

We need to add $\mathbf{c}_I^T$ to the reduced cost coefficients in the columns of the initial identity matrix to obtain a solution to the dual problem.

Two cases may be distinguished:

- If the columns of matrix $\mathbf{I}$ correspond to slack variables, then $\mathbf{c}_I = \mathbf{0}$ holds.

- If there are artificial variables in the columns corresponding to the initial matrix $\mathbf{I}$, then the cost coefficients of the artificial variables are M, and they appear in vector $\mathbf{c}_I$, due to the penalty of the objective function.

5. Economic interpretation of duality

5.1 Shadow prices

A linear model and its optimal basis $\mathbf{B}$, $\mathbf{x}^*$ and z^*. The dual optimal solution $\mathbf{y}^*$ and G^*.

If the right-hand-side vector $\mathbf{b}$ changes to $\mathbf{b} + \triangle\mathbf{b}$, assuming that the current basis remains optimal,

- $\overset{\wedge}{\mathbf{x}}_B = \mathbf{B}^{-1}(\mathbf{b} + \triangle\mathbf{b}) = \mathbf{x}_B + \mathbf{B}^{-1}\triangle\mathbf{b}$.

- $z_j - c_j = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{a}_j - c_j$.

- $\hat{G}{}^* = \mathbf{y}^{*T}(\mathbf{b} + \triangle\mathbf{b}) = \mathbf{y}^{*T}\mathbf{b} + \mathbf{y}^{*T}\triangle\mathbf{b} = G^* + \mathbf{y}^{*T}\triangle\mathbf{b} = z^* + \mathbf{y}^{*T}\triangle\mathbf{b}$.

Assume that $\triangle b_i = 1$ and all the rest are zero. The optimal value of the dual variable y_i^* is the rate of change (increase) of the optimal objective value.

$$\mathbf{y}^{*T}\triangle\mathbf{b} = (y_1^*, \ \ldots, \ y_i^*, \ \ldots, \ y_m^*) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = y_i^*.$$

Definition 3 (Shadow price) *The optimal dual variable y_i^* is said to be the shadow price of the ith right-hand-side value, $i = 1, \ldots, m$, if increasing/decreasing a unit in the ith right-hand-side value, and keeping the rest of the right-hand-side values unchanged, does not violate the optimality of the tableau.*

5.2 The economic cost of the primal variables and the interpretation of the simplex method

Example.

	Products				Resource
Resource	1	2	3	4	available
A	2	3	$\frac{3}{2}$	4	300
B	2	4	3	1	500
C	5	1	2	2	250
Profit	4	3	6	2	

x_j: the units of product j produced, $j = 1, 2, 3, 4$.

y_1, y_2, y_3: the price to pay for a unit of each of the resources A, B and C, respectively.

max $z = 4x_1 + 3x_2 + 6x_3 + 2x_4$
  subject to
  $2x_1 + 3x_2 + \frac{3}{2}x_3 + 4x_4 \leq 300$
  $2x_1 + 4x_2 + 3x_3 + x_4 \leq 500$
  $5x_1 + x_2 + 2x_3 + 2x_4 \leq 250$
  $x_1, x_2, x_3, x_4 \geq 0$

min $G = 300y_1 + 500y_2 + 250y_3$
  subject to
  $2y_1 + 2y_2 + 5y_3 \geq 4$
  $3y_1 + 4y_2 + y_3 \geq 3$
  $\frac{3}{2}y_1 + 3y_2 + 2y_3 \geq 6$
  $4y_1 + y_2 + 2y_3 \geq 2$
  $y_1, y_2, y_3 \geq 0$

$$z_j - c_j = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{a}_j - c_j = \mathbf{y}^T \mathbf{a}_j - c_j = \sum_{i=1}^{m} a_{ij} y_i - c_j.$$

$z_j - c_j < 0$ is the jth dual constraint $\sum_{i=1}^{m} a_{ij} y_i < c_j$, which means that when $z_j - c_j < 0$, the economic cost of the production of j is lower than the profit c_j associated to its production, and thus, the production of j becomes profitable.

6. The dual simplex method

Write the model in maximization symmetric form. Basis $\mathbf{B} = \mathbf{I}$ formed by slack variables.

6.1 The dual simplex algorithm

The objective is to maximize.

* **Step 1.** Construct the initial tableau, where $z_j - c_j \geq 0$ for all j.

* **Step 2.** With regard to primal feasibility, there are two cases to consider.

  - If $x_{Bi} \geq 0$, $i = 1, \ldots, m$, the current solution is optimal. Stop.

  - If there exists $x_{Bi} < 0$, then the dual solution may be improved. Go to Step 3.

* **Step 3.** Basis modification.

  - Leaving vector: $\mathbf{a}_r$. Pivot row: the rth row.

  $$x_{Br} = \min_i \{ x_{Bi} \ / \ x_{Bi} < 0 \}.$$

  - Entering vector: $\mathbf{a}_k$. Pivot column: the kth.

  $$\frac{z_k - c_k}{y_{rk}} = \max_j \left\{ \frac{z_j - c_j}{y_{rj}} \ / \ y_{rj} < 0 \right\}.$$

  y_{rk} is the pivot element.

  If $y_{rj} \geq 0$ for all j , the problem is infeasible. Stop.

* **Step 4.** Compute the new tableau by pivoting in the same way as stated in the simplex algorithm. Go to Step 2.
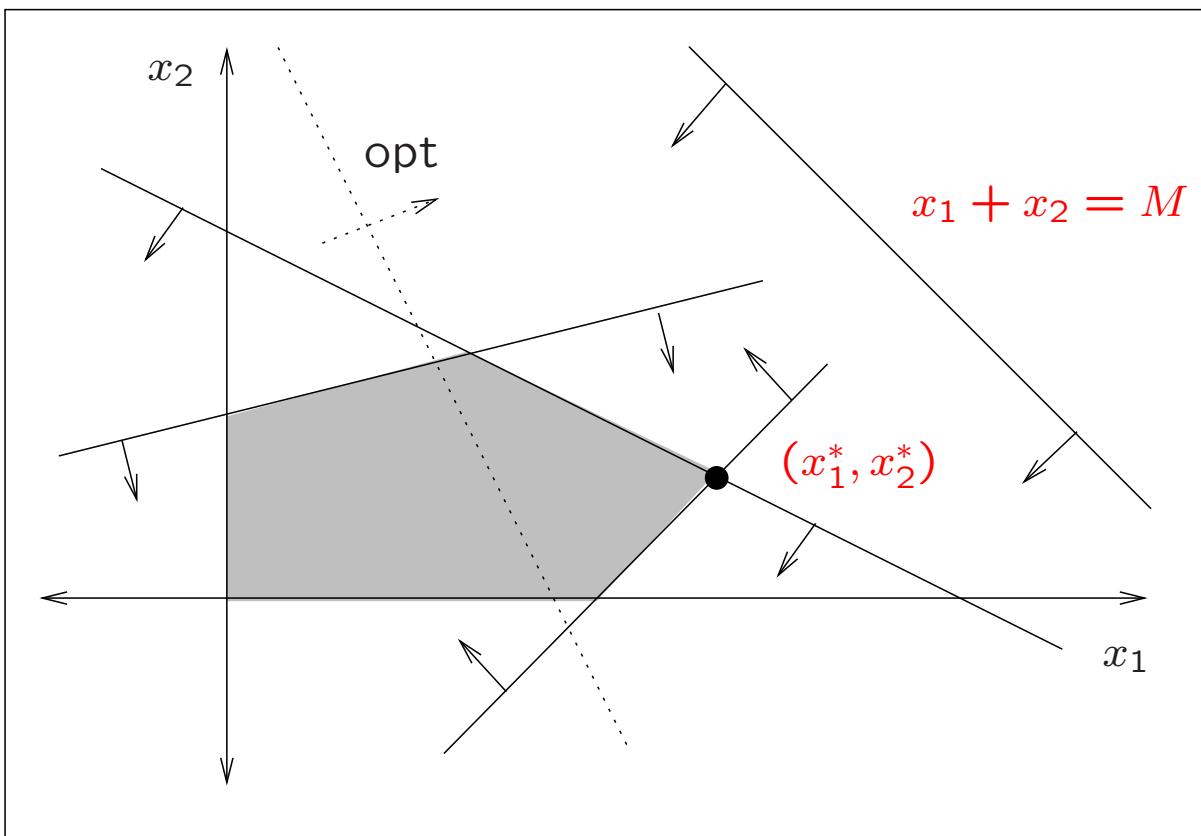
7. The artificial constraint technique

When the initial tableau is not dual feasible, an artificial constraint is added to the model:

$$\sum_{j\in N} x_j \leq M,$$

where N is the set of variables such that the reduced cost coefficients $z_j - c_j$ are negative in the initial tableau.

7.1 The effect of the artificial constraint

The introduction of the artificial constraint to the model should never modify the feasible region of the original model. M must be positive and as large as necessary.

If M is not large enough, some points in the feasible region are discarded, because the artificial constraint eliminates them.
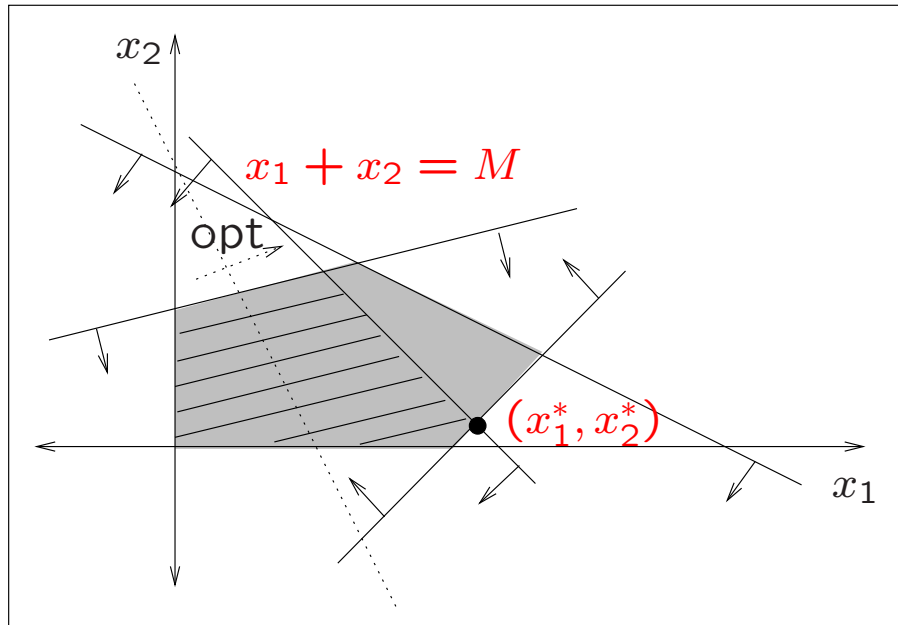
When the feasible region of the problem is unbounded, the artificial constraint bounds it, even if M is large.
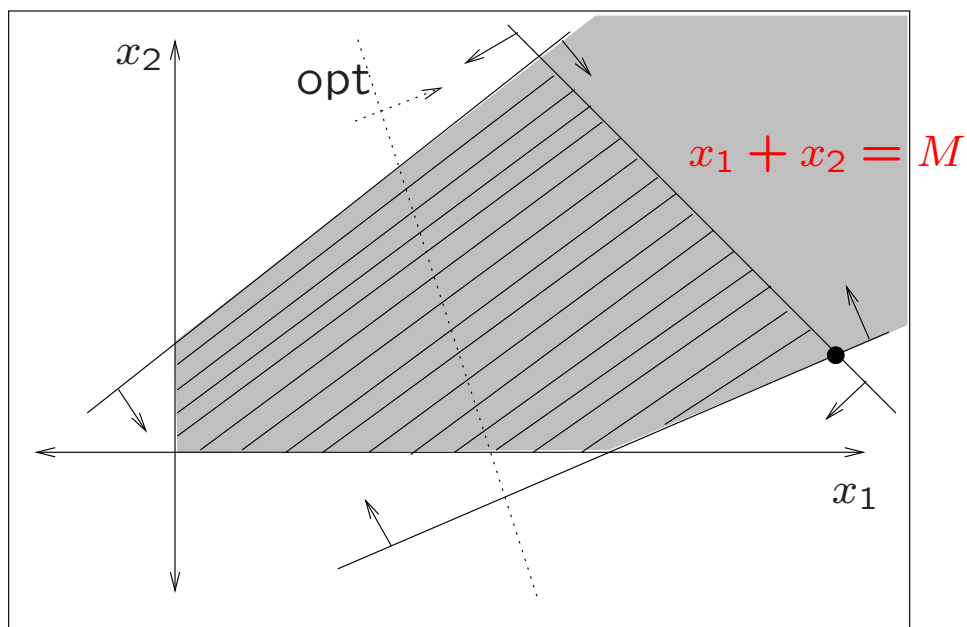
7.2 The dual simplex algorithm with artificial constraint

Write the model in maximization symmetric form. Basis $\mathbf{B} = \mathbf{I}$ formed by slack variables. The objective is to maximize.

* **Step 1.** Construct the initial tableau.

* **Step 2.** With regard to the dual feasibility,

  * If $z_j - c_j \geq 0$ holds for all j, then go to Step 3.

  * If there exists at least a column such that $z_j - c_j < 0$, then add an artificial constraint to the model and construct the initial tableau for the model with the additional constraint. The entering vector $\mathbf{a}_k$ is selected as follows:

    $$z_k - c_k = \min_j \{ z_j - c_j \; / \; z_j - c_j < 0 \}.$$

    Column k is the pivot column. Select the slack vector of the artificial constraint to leave the basis; the corresponding row is the pivot row. Compute the new tableau by pivoting in the same way as stated in the simplex algorithm. The new tableau is dual feasible. Go to Step 3.

* **Step 3.** Primal feasibility.

  * If the model has no additional artificial constraint,

    * If $x_{Bi} \geq 0$ for all i, then the solution is optimal. Stop.

    * If there exists at least a row i such that $x_{Bi} < 0$, then the solution may be improved. Go to Step 4.

- If the model has an additional artificial constraint,

  * If $x_{Bi} \geq 0$ for all i, and the slack variable of the artificial constraint is basic and positive, then the solution is optimal. Stop.

  * If $x_{Bi} \geq 0$ for all i, and the slack variable of the artificial constraint is nonbasic, or it is basic and has value zero, then the problem is unbounded. Stop.

  * If there exists at least a row i such that $x_{Bi} < 0$, then the solution may be improved. Go to Step 4.

* **Step 4.** Basis modification.

  - Select a vector $\mathbf{a}_r$ to leave the basis:

  $$x_{Br} \;=\; \min_i \; \{ \; x_{Bi} \; / \; x_{Bi} < 0 \; \}.$$

  The rth row is the pivot row.

  - Select a vector $\mathbf{a}_k$ to enter the basis:

  $$\frac{z_k - c_k}{y_{rk}} \;=\; \max_j \; \left\{ \frac{z_j - c_j}{y_{rj}} \; / \; y_{rj} < 0 \; \right\}.$$

  The kth column is the pivot column. y_{rk} is the pivot element. Go to Step 5.
  If $y_{rj} \geq 0$ for all j in the pivot row, the problem is infeasible. Stop.

* **Step 5.** Compute the new tableau by pivoting in the same way as stated in the simplex algorithm. Go to Step 3.