1.LINEAR MODELING. GRAPHICAL SOLUTION

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1. The linear model

A linear model deals with optimizing a linear function with several variables, given certain linear constraint inequalities.

$$\text{opt} \quad z = \mathbf{c}^T\mathbf{x} \tag{1}$$

subject to

$$\mathbf{Ax} \begin{matrix} \le \\ = \\ \ge \end{matrix} \mathbf{b} \tag{2}$$

$$\mathbf{x} \ge \mathbf{0} \tag{3}$$

where,

(1) is the objective function,

(2) are the constraints and

(3) are the nonnegativity constraints.

The elements that appear in the model are:

- $\mathbf{x}$: is the vector of decision variables.
- $\mathbf{c}^T$: is the vector of cost coefficients.
- $\mathbf{b}$: is the right-hand-side vector.
- $\mathbf{A}$: is the constraint matrix.

$\mathbf{c}^T$, $\mathbf{b}$ and $\mathbf{A}$ are known parameters; $\mathbf{x}$ contains the variables whose values have to be determined.

2. Notation

1.

$$
\begin{aligned}
\text{opt } z &= c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} & \\
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\begin{smallmatrix}\le\\=\\\ge\end{smallmatrix} b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\begin{smallmatrix}\le\\=\\\ge\end{smallmatrix} b_2 \\
\vdots \qquad \vdots \qquad \ddots \qquad \vdots \quad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\begin{smallmatrix}\le\\=\\\ge\end{smallmatrix} b_m \\
x_1, x_2, \ldots, x_n &\ge 0
\end{aligned}
$$

2.

$$
\text{opt } z = (c_1, \ldots, c_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}
$$

subject to

$$
\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix}\le\\=\\\ge\end{matrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}
$$

$$
(x_1, x_2, \cdots, x_n)^T \ge (0, 0, \cdots, 0)^T
$$

3.

$$
\begin{aligned}
\text{opt } z &= c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} & \\
\mathbf{a}_1x_1 + \mathbf{a}_2x_2 + \cdots + \mathbf{a}_nx_n &\begin{smallmatrix}\le\\=\\\ge\end{smallmatrix} \mathbf{b} \\
x_j \ge 0 \ , \ j &= 1, \ldots, n
\end{aligned}
$$

3. Linear programming modeling

3.1 A transportation problem

A company produces bicycles at three plants in cities C_1, C_2 and C_3. Their production capacity is 1000, 2100 and 1500 bicycles per month, respectively. Four customers, A, B, C and D, from four different locations are demanding 800, 1100, 900 and 1300 bicycles, respectively, every month. The unit costs of transporting a bicycle from a given city to a given customer:

	A	B	C	D
C_1	10	8	10	13
C_2	19	6	15	16
C_3	14	8	9	6

The minimum-cost shipping must be found.

Decision variables.

x_{ij}: number of bicycles transported monthly from city C_i to customer j, $i = 1, 2, 3$, $j = A, B, C, D$.

Linear model.

$$\begin{aligned}
\min \quad z = {} & 10x_{1A} + 8x_{1B} + 10x_{1C} + 13x_{1D} + 19x_{2A} + 6x_{2B} + \\
& 15x_{2C} + 16x_{2D} + 14x_{3A} + 8x_{3B} + 9x_{3C} + 6x_{3D} \\
\text{subject to} & \\
& x_{1A} + x_{1B} + x_{1C} + x_{1D} \leq 1000 \\
& x_{2A} + x_{2B} + x_{2C} + x_{2D} \leq 2100 \\
& x_{3A} + x_{3B} + x_{3C} + x_{3D} \leq 1500 \\
& x_{1A} + x_{2A} + x_{3A} \geq 800 \\
& x_{1B} + x_{2B} + x_{3B} \geq 1100 \\
& x_{1C} + x_{2C} + x_{3C} \geq 900 \\
& x_{1D} + x_{2D} + x_{3D} \geq 1300 \\
& x_{ij} \geq 0, \quad i = 1, 2, 3, \quad j = A, B, C, D
\end{aligned}$$

3.2 A production problem

Pieces P_1, P_2 and P_3 are manufactured by using machines A, B and C. The number of hours each machine is available for manufacturing and the production cost:

Machine	Availability (hours/week)	Production cost (euro/hour)
A	1000	6
B	1000	4
C	1000	5

Each type of piece needs a different amount of processing time in each of the machines:

Machine	P_1	P_2	P_3
A	1	2	3
B	2	3	1
C	1	1	1

The availability of materials M_1 and M_2 used in the production process: 1000 kg and 1200 kg, respectively. The amount of material needed in the production of one piece of each type:

Piece	M_1 (kg/piece)	M_2 (kg/piece)
P_1	1	2
P_2	1	3
P_3	3	1

1 kg of material M_1 costs 1.5 euros and 1 kg of material M_2 3 euros. Each piece is sold at the price of 50, 56 and 70 euros, respectively.

The firm aims to find the maximum benefit production.

Decision variables.

x_j: number of pieces P_j that the firm will produce weekly, $j = 1, 2, 3$.

Objective function. To maximize benefit.

Benefit = Selling price − Materials cost − Production cost.

* Selling price = $50x_1 + 56x_2 + 70x_3$

* Materials cost = $(1 \times 1.5 + 2 \times 3)x_1 + (1 \times 1.5 + 3 \times 3)x_2 + (3 \times 1.5 + 1 \times 3)x_3$

* Production cost = $(1 \times 6 + 2 \times 4 + 1 \times 5)x_1 + (2 \times 6 + 3 \times 4 + 1 \times 5)x_2 + (3 \times 6 + 1 \times 4 + 1 \times 5)x_3$

Linear model.

$$
\begin{aligned}
\max \ z = 23.5x_1 + 16.5x_2 + 35.5x_3 \\
\text{subject to} \\
x_1 + 2x_2 + 3x_3 \leq 1000 \\
2x_1 + 3x_2 + x_3 \leq 1000 \\
x_1 + x_2 + x_3 \leq 1000 \\
x_1 + x_2 + 3x_3 \leq 1000 \\
2x_1 + 3x_2 + x_3 \leq 1200 \\
x_1, x_2, x_3 \geq 0
\end{aligned}
$$

3.3 A product-mix problem

A fuel company produces two types of fuel, A and B by mixing three types of crude oil.

The following table shows the number of crude oil barrels available and the cost of each barrel:

	Barrels available	Cost
Crude oil O_1	2000	10
Crude oil O_2	3000	8
Crude oil O_3	1000	12

The quality of fuels A and B is considered to be acceptable if:

- At least 30% of fuel A is crude oil O_1, at least 20% crude oil O_2 and no more than 30% crude oil O_3.

- At least 25% of the composition of fuel B must be crude oil O_1, at least 25% crude oil O_2, and at least 25% crude oil O_3.

The selling prices of a barrel of fuel A and fuel B are 40 and 35 units, respectively.

The aim is to organize the fuel production in order to obtain the maximum benefit.

Decision variables.

x_{ij} : The amount of barrels of crude oil O_i in the composition of fuel j, $i = 1, 2, 3$, $j = A, B$.

Objective function. To maximize benefit: "Selling price (SP)" − "Production cost (PC)".

SP $= 40(x_{1A} + x_{2A} + x_{3A}) + 35(x_{1B} + x_{2B} + x_{3B})$

PC $= 10(x_{1A} + x_{1B}) + 8(x_{2A} + x_{2B}) + 12(x_{3A} + x_{3B})$

Linear model.

$$\max\ z = 30x_{1A} + 32x_{2A} + 28x_{3A} + 25x_{1B} + 27x_{2B} + 23x_{3B}$$

subject to

$$\begin{aligned}
x_{1A} + x_{1B} &\leq 2000 \\
x_{2A} + x_{2B} &\leq 3000 \\
x_{3A} + x_{3B} &\leq 1000 \\
x_{1A} &\geq \frac{30}{100}(x_{1A} + x_{2A} + x_{3A}) \\
x_{2A} &\geq \frac{20}{100}(x_{1A} + x_{2A} + x_{3A}) \\
x_{3A} &\leq \frac{30}{100}(x_{1A} + x_{2A} + x_{3A}) \\
x_{1B} &\geq \frac{25}{100}(x_{1B} + x_{2B} + x_{3B}) \\
x_{2B} &\geq \frac{25}{100}(x_{1B} + x_{2B} + x_{3B}) \\
x_{3B} &\geq \frac{25}{100}(x_{1B} + x_{2B} + x_{3B}) \\
x_{ij} \geq 0, \quad i &= 1, 2, 3, \quad j = A, B
\end{aligned}$$

3.4 A diet problem

Diet: at least 25 milligrams of vitamin A, between 25 and 30 milligrams of vitamin B, at least 22 milligrams of vitamin C and no more than 17 milligrams of vitamin D.

Food	Vitamins (mg/g)				Cost
	A	B	C	D	(euro/g)
F_1	2	1	0	1	0.014
F_2	1	2	1	2	0.009
F_3	1	0	2	0	0.013
F_4	1	2	1	1	0.016

The minimum cost diet which satisfies the nutritional requirements must be found.

Decision variables.

x_j: grams of each type of food F_j included in the diet, $j = 1, 2, 3, 4$.

Linear model.

$$\min \; z = 0.014x_1 + 0.009x_2 + 0.013x_3 + 0.016x_4$$

subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 &\geq 25 \\ x_1 + 2x_2 + 2x_4 &\geq 25 \\ x_1 + 2x_2 + 2x_4 &\leq 30 \\ x_2 + 2x_3 + x_4 &\geq 22 \\ x_1 + 2x_2 + x_4 &\leq 17 \\ x_j \geq 0, \;\; j &= 1, 2, 3, 4 \end{aligned}$$

3.5 A cutting problem

5m long wooden sticks must be cut. Demands: 100 3m long sticks, 100 2m long sticks, 300 1.5m long ones and 150 1m long sticks. The enterprise wants to minimize the waste incurred in meeting the customer demands.

7 different ways to cut the sticks:

Cutting option	Length 3m	2m	1.5m	1m
1	1	1	0	0
2	1	0	0	2
3	0	2	0	1
4	0	1	2	0
5	0	1	0	3
6	0	0	2	2
7	0	0	0	5

Decision variables.

x_j: number of 5m long sticks cut according to cutting option j, $j = 1, \ldots, 7$.

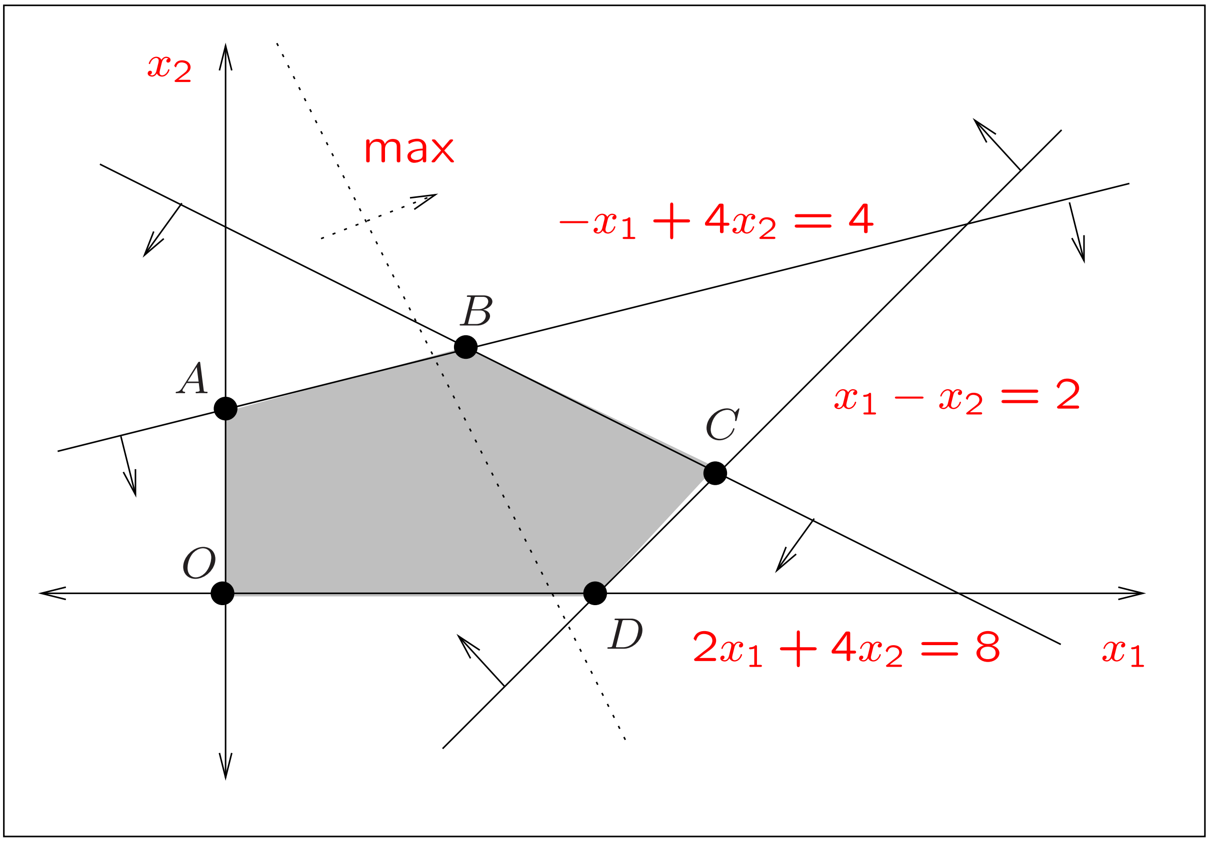
Linear model.

$$\begin{aligned} \text{min} \quad & z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\ \text{subject to} \quad & \\ & x_1 + x_2 \geq 100 \\ & x_1 + 2x_3 + x_4 + x_5 \geq 100 \\ & 2x_4 + 2x_6 \geq 300 \\ & 2x_2 + x_3 + 3x_5 + 2x_6 + 5x_7 \geq 150 \\ & x_j \geq 0, \ j = 1, \ldots, 7 \end{aligned}$$

4. Graphical solution

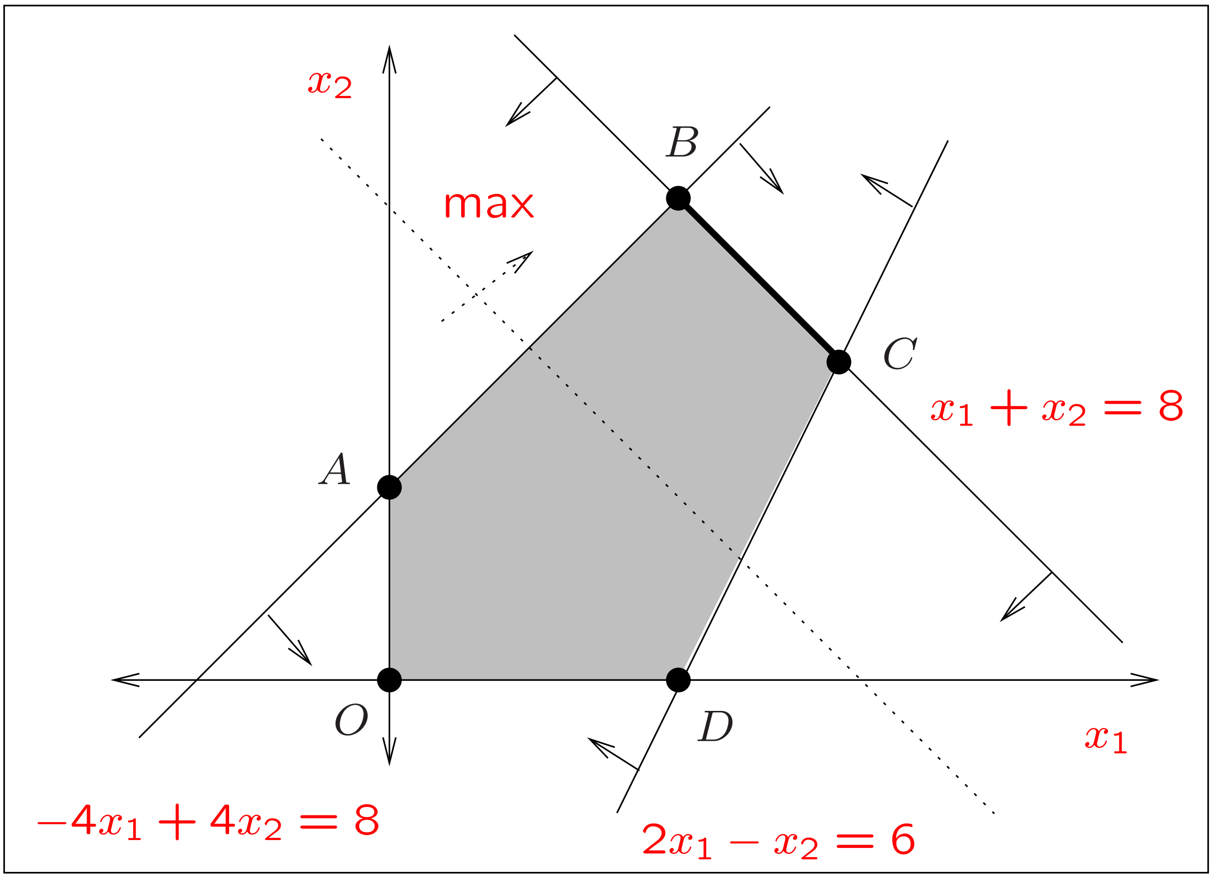
4.1 A problem with a unique optimal solution

$$\begin{aligned} \max \; z = 6x_1 + 3x_2 & \\ \text{subject to} & \\ 2x_1 + 4x_2 &\leq 8 \\ -x_1 + 4x_2 &\leq 4 \\ x_1 - x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$



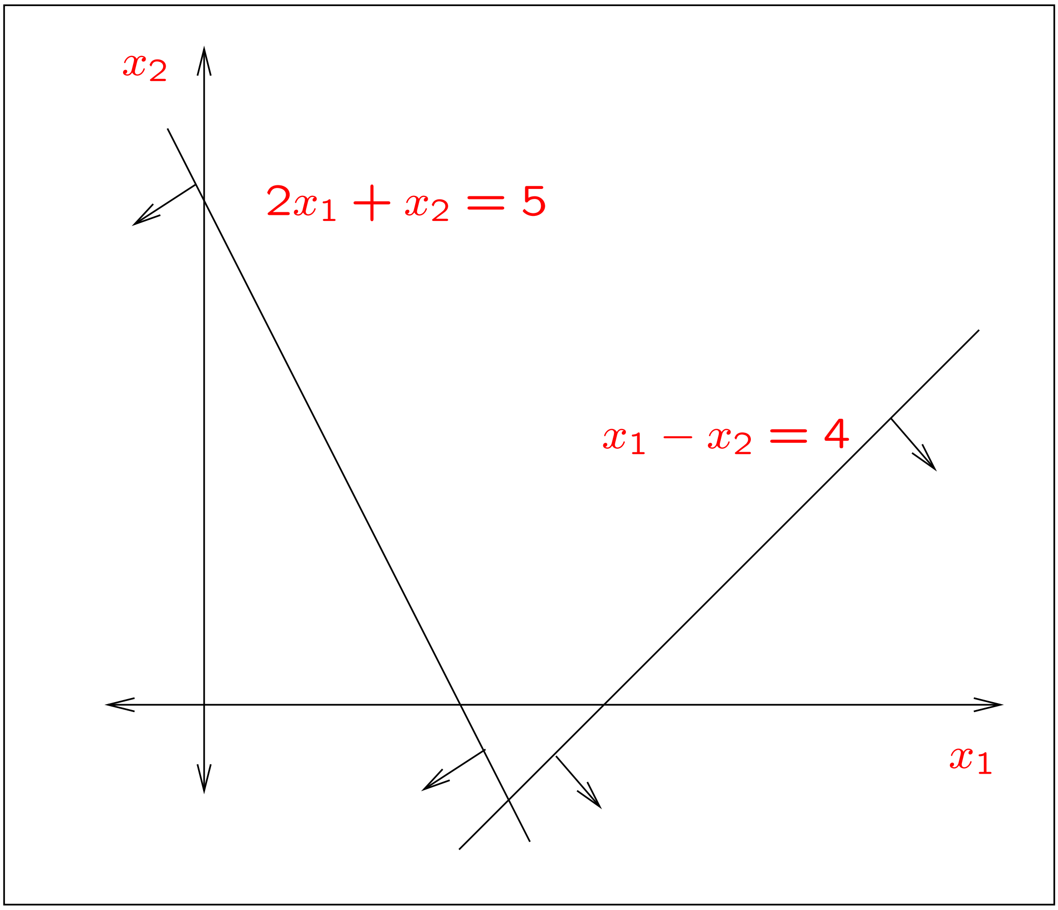
4.2 A problem with multiple optimal solutions

$$\begin{aligned} \max\ \ z = x_1 + x_2 \\ \text{subject to} \\ x_1 + x_2 \leq 8 \\ -4x_1 + 4x_2 \leq 8 \\ 2x_1 - x_2 \leq 6 \\ x_1, x_2 \geq 0 \end{aligned}$$



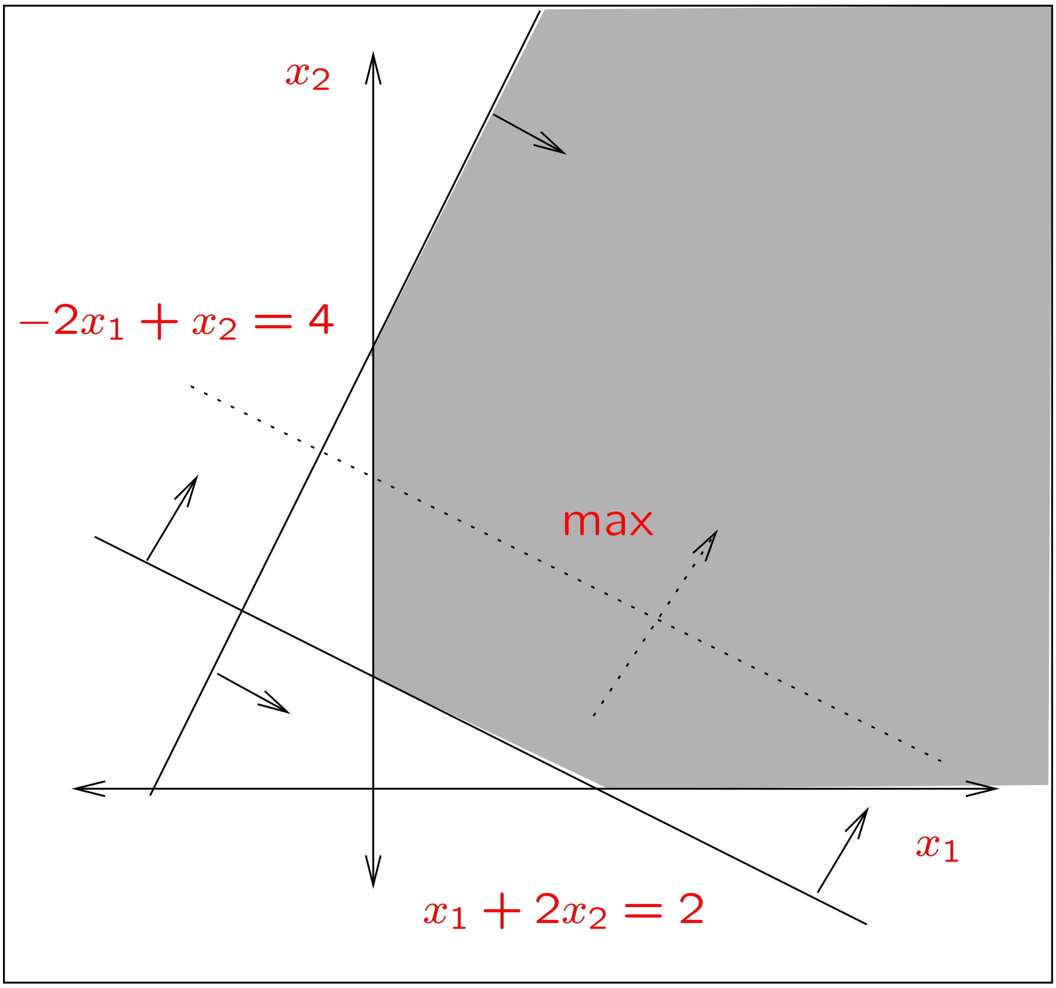
4.3 An infeasible problem

$$\begin{aligned} \max \;\; z = x_1 + x_2 \\ \text{subject to} \\ 2x_1 + x_2 \leq 5 \\ x_1 - x_2 \geq 4 \\ x_1, x_2 \geq 0 \end{aligned}$$



4.4 An unbounded feasible region. Unbounded solution

$$\max \; z = x_1 + 2x_2$$
$$\text{subject to}$$
$$x_1 + 2x_2 \geq 2$$
$$-2x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$



4.5 An unbounded region. A bounded solution

$$\begin{aligned} \min \ z = x_1 + 2x_2 \\ \text{subject to} \\ x_1 + 2x_2 \geq 2 \\ -2x_1 + x_2 \leq 4 \\ x_1, x_2 \geq 0 \end{aligned}$$