

Chapter 4

Sensitivity analysis

The sensitivity analysis is performed after a given linear problem has been solved, with the aim of studying how changes to the problem affect the optimal solution. In particular, the following discrete changes to the problem are considered: a change in the cost coefficients vector \mathbf{c} , a change in the right-hand-side vector \mathbf{b} and a change in the constraint matrix \mathbf{A} . The addition of a new variable to the model, and the addition of a new constraint are also analyzed.

The sensitivity analysis takes advantage of the calculations performed in solving the original problem, because the analysis starts at the current optimal tableau.

The analysis of discrete changes can be extended to study the effect that continuous changes have on the optimal solution. This kind of analysis is called *parametric analysis*.

4.1 The original problem and the optimal tableau

In this chapter we examine the sensitivity analysis for each one of the discrete changes mentioned in the introduction; the effect of two or more simultaneous changes can also be analyzed with the method discussed here.

Let us consider the following linear model:

$$\begin{aligned} \max z &= \mathbf{c}^T \mathbf{x} \\ \text{subject to} \\ \mathbf{Ax} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

Assuming that all components of vector \mathbf{b} are greater than or equal to zero, the standard form of the model is obtained by adding a slack variable to each of the constraints.

$$\begin{aligned} \max z &= \mathbf{c}^T \mathbf{x} + \mathbf{0}^T \mathbf{x}_s \\ \text{subject to} \\ \mathbf{Ax} + \mathbf{Ix}_s &= \mathbf{b} \\ \mathbf{x}, \mathbf{x}_s &\geq \mathbf{0} \end{aligned}$$

As we have seen in previous chapters, if the canonical basis is chosen to be the initial basis, then all the computations needed to construct the initial tableau are directly obtained from the linear model. In the linear model stated above, the initial canonical basis consists of slack variables.

The inverse basis matrix \mathbf{B}^{-1} can be extracted from the simplex tableau as follows: it is the submatrix in rows 1 through m under the original identity columns. This fact is specially important in the application of the sensitivity analysis, because there is no need to compute \mathbf{B}^{-1} , which is always in the optimal tableau.

- **The initial tableau.**

		Original variables	Slack variables	
		$x_1 \dots x_n$	$x_{n+1} \dots x_{n+m}$	
		$-\mathbf{c}^T$	$\mathbf{0}$	0
B		A	I	b

- **The optimal tableau.**

		Original variables	Slack variables	
		$x_1 \dots x_n$	$x_{n+1} \dots x_{n+m}$	
		$\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A} - \mathbf{c}^T$	$\mathbf{c}_B^T \mathbf{B}^{-1}$	$z = \mathbf{c}_B^T \mathbf{x}_B$
B		$\mathbf{B}^{-1} \mathbf{A}$	\mathbf{B}^{-1}	$\mathbf{x}_B = \mathbf{B}^{-1} \mathbf{b}$

The optimal tableau for a linear model is primal feasible, because $\mathbf{x}_B \geq 0$. It is also dual feasible, because $z_j - c_j \geq 0$ for any j . As we have just said, the sensitivity analysis is based on the use of the optimal tableau. If we look at it, we can see that the effect of each of the discrete changes analyzed by the sensitivity analysis will occur in a specific part of the tableau. For instance, a change in the right-hand-side vector \mathbf{b} affects the \mathbf{x}_B , which will have to be updated with the consequence that the primal feasibility may be violated. Similarly, the effect of a change in the cost coefficients vector \mathbf{c} will occur on row zero of the tableau, and consequently the dual feasibility may be violated. In the case that either the primal or the dual feasibility are violated, the dual simplex algorithm or the simplex algorithm will be used to find the new optimal solution.

4.2 Example

We introduce a production problem and refer back to it along the chapter, to interpret the effect of each one of the discrete changes analyzed.

An enterprise uses resources 1 and 2 in the production of products A , B and C . The necessary amount of resource 1 and 2 in the production of a unit of each type of product, the amount of resource available and the benefit obtained from the production of a unit of each type of product are given in Table 4.1.

The following linear model is formulated to maximize the benefit obtained from the production. The necessary slack variables have been introduced to obtain the maximization standard form of the model.

$$\max z = 3x_1 + 2x_2 + x_3 + 0x_4 + 0x_5$$

subject to

Resources	Products			Resource availability
	A	B	C	
1	4	2	3	40
2	2	2	1	30
Benefit	3	2	1	

Table 4.1: Resources and benefit

$$\begin{aligned}
 4x_1 + 2x_2 + 3x_3 + x_4 &= 40 \\
 2x_1 + 2x_2 + x_3 + x_5 &= 30 \\
 x_1, \dots, x_5 &\geq 0
 \end{aligned}$$

Let x_1, x_2 and x_3 be the number of units of product A, B and C to be produced. The associated optimal tableau is:

	x_1	x_2	x_3	x_4	x_5	
	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	35
\mathbf{a}_1	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	5
\mathbf{a}_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	10

The optimal solution is $x_1^* = 5$, $x_2^* = 10$ and $x_3^* = 0$, and the maximum benefit $z^* = 35$. This maximum benefit is obtained by producing 5 units of product A and 10 units of product B , thus, using up all the available resource units in the production of A and B . \square

4.3 Change in vector \mathbf{b}

Given a linear model and its optimal tableau, let us suppose that the right-hand-side vector changes from \mathbf{b} to $\hat{\mathbf{b}}$. Therefore, we have the original model (Model 1) and a perturbation of it (Model 2).

<u>Model 1</u>	<u>Model 2</u>
$\max z = \mathbf{c}^T \mathbf{x}$	$\max z = \mathbf{c}^T \mathbf{x}$
subject to	subject to
$\mathbf{Ax} \leq \mathbf{b}$	$\mathbf{Ax} \leq \hat{\mathbf{b}}$
$\mathbf{x} \geq \mathbf{0}$	$\mathbf{x} \geq \mathbf{0}$

Given that \mathbf{b} is not used to calculate the reduced cost coefficients of row zero, changes in \mathbf{b} leave row zero unaffected. However, the last column of the tableau changes (see the shaded column in Figure 4.1), and after updating it, the new tableau may become primal infeasible.

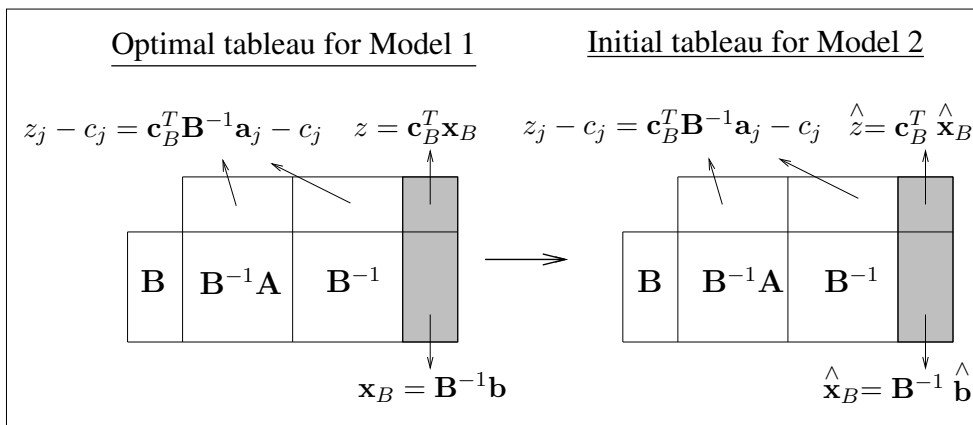


Figure 4.1: Changing vector \mathbf{b}

The change in vector \mathbf{b} affects the primal solution and the objective value, which will have to be updated.

$$\hat{\mathbf{x}}_B = \mathbf{B}^{-1} \hat{\mathbf{b}}$$

$$\hat{z} = \mathbf{c}_B^T \hat{\mathbf{x}}_B$$

Once the last column of the tableau has been updated, we have the initial tableau for Model 2. The following two cases can arise:

- **Case 1.** If $\hat{\mathbf{x}}_B \geq \mathbf{0}$, the tableau is primal feasible, and thus, the current basis \mathbf{B} remains optimal. Therefore, the initial tableau for Model 2 is optimal,

and the optimal solution and the optimal objective value are $\hat{\mathbf{x}}_B$ and \hat{z} , respectively.

- **Case 2.** If $\hat{\mathbf{x}}_B \not\geq \mathbf{0}$, the tableau is not primal feasible. The dual simplex algorithm will be applied to restore feasibility, starting at the initial tableau for Model 2.

Example. Consider the production problem of Section 4.2 and its corresponding optimal tableau. Let us analyze the following two changes in the right-hand-side vector \mathbf{b} .

First change. Let us suppose that vector $\mathbf{b}^T = (40, 30)$ is replaced by $\hat{\mathbf{b}}^T = (38, 36)$. Let us update the last column of the tableau.

$$\hat{\mathbf{x}}_B = \mathbf{B}^{-1} \hat{\mathbf{b}} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 38 \\ 36 \end{pmatrix} = \begin{pmatrix} 1 \\ 17 \end{pmatrix} \geq \mathbf{0}.$$

$$\hat{z} = \mathbf{c}_B^T \hat{\mathbf{x}}_B = (3, 2) \begin{pmatrix} 1 \\ 17 \end{pmatrix} = 37.$$

We find the following initial tableau for Model 2:

	x_1	x_2	x_3	x_4	x_5	
	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	37
\mathbf{a}_1	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	1
\mathbf{a}_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	17

The primal feasibility is kept, and thus the basis \mathbf{B} remains optimal. However, the optimal solution and the optimal value have been updated: $x_1^* = 1, x_2^* = 17, x_3^* = 0, z^* = 37$. The fact that the optimal basis remains unchanged can be interpreted in the following way: the production of A and B should continue, even though they will be produced in different quantities.

Second change. Let us suppose that vector $\mathbf{b}^T = (40, 30)$ is replaced by $\hat{\mathbf{b}}^T = (20, 60)$. Let us update the last column of the tableau.

$$\hat{\mathbf{x}}_B = \mathbf{B}^{-1} \hat{\mathbf{b}} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 20 \\ 60 \end{pmatrix} = \begin{pmatrix} -20 \\ 50 \end{pmatrix} \not\geq \mathbf{0}.$$

$$\hat{z} = \mathbf{c}_B^T \hat{\mathbf{x}}_B = (3, 2) \begin{pmatrix} -20 \\ 50 \end{pmatrix} = 40.$$

The tableau is not primal feasible, and thus, the initial tableau for Model 2 is not optimal. We replace $\hat{\mathbf{x}}_B$ and \hat{z} and apply the dual simplex algorithm to restore feasibility.

		x_1	x_2	x_3	x_4	x_5	
		0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	40
3	\mathbf{a}_1	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	-20
2	\mathbf{a}_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	50
		1	0	2	1	0	20
0	\mathbf{a}_5	-2	0	-2	-1	1	40
2	\mathbf{a}_2	2	1	$\frac{3}{2}$	$\frac{1}{2}$	0	10

The optimal solution to Model 2 is $x_1^* = 0$, $x_2^* = 10$, $x_3^* = 0$, and the optimal value $z^* = 20$. Note that the primal feasibility loss has made the optimal basis change. Now, the optimal solution indicates that the production of B should continue (vector \mathbf{a}_2 remains in the optimal basis), but the change in vector \mathbf{b} has made product A unprofitable (vector \mathbf{a}_1 is no longer in the optimal basis). \square

4.4 Change in vector \mathbf{c}

Given a linear model and its optimal tableau, let us suppose the cost coefficients vector \mathbf{c} changes to $\hat{\mathbf{c}}$. Therefore, we have the original model (Model 1) and a perturbation of it (Model 2).

Model 1	Model 2
$\max z = \mathbf{c}^T \mathbf{x}$	$\max z = \hat{\mathbf{c}}^T \mathbf{x}$
subject to	subject to
$\mathbf{Ax} \leq \mathbf{b}$	$\mathbf{Ax} \leq \mathbf{b}$
$\mathbf{x} \geq \mathbf{0}$	$\mathbf{x} \geq \mathbf{0}$

The change in vector \mathbf{c} affects the reduced cost coefficients $z_j - c_j$ of row zero and the objective value z (see the shaded row in Figure 4.2). Therefore, we need to update row zero in the optimal tableau for Model 1 to study the change. The updating of row zero may result in a nonoptimal tableau (dual infeasible).

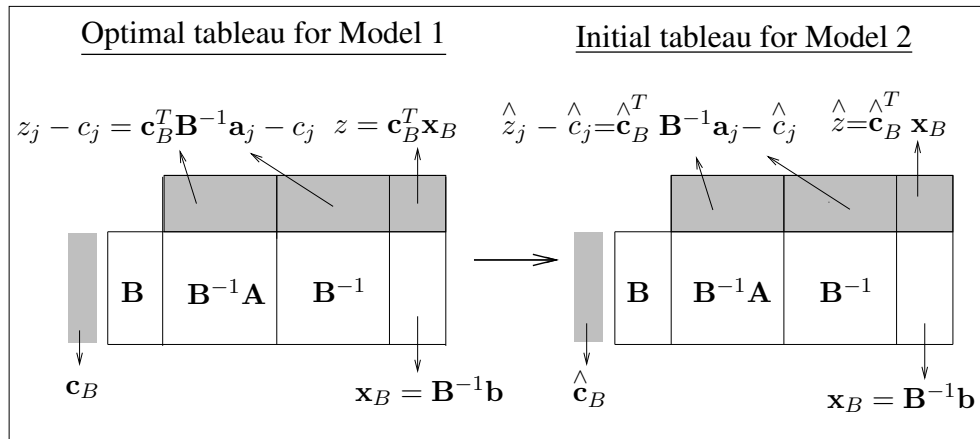


Figure 4.2: Change in vector \mathbf{c} .

As a consequence of the change in vector \mathbf{c} , we need to update the following:

$$\hat{z}_j - \hat{c}_j = \hat{\mathbf{c}}_B^T \mathbf{B}^{-1} \mathbf{a}_j - \hat{c}_j = \hat{\mathbf{c}}_B^T \mathbf{y}_j - \hat{c}_j.$$

$$\hat{z} = \hat{\mathbf{c}}_B^T \mathbf{x}_B.$$

Once $\hat{z}_j - \hat{c}_j$ and \hat{z} are computed, they are replaced in the tableau and we find the initial tableau for Model 2. The following two cases can arise:

- **Case 1.** If $\hat{z}_j - \hat{c}_j \geq 0$ for any j , the tableau is dual feasible, and thus, the current basis \mathbf{B} remains optimal. Therefore, the initial tableau for Model 2 is optimal; \mathbf{x}_B remains optimal, and the optimal objective value is $\hat{z} = \hat{\mathbf{c}}_B^T \mathbf{x}_B$.
- **Case 2.** If there exists any $\hat{z}_j - \hat{c}_j < 0$, the current basis \mathbf{B} is no longer optimal since the tableau is not dual feasible. The simplex algorithm will be applied to restore feasibility, starting at the initial tableau for Model 2.

Example. To illustrate the preceding ideas, consider the production problem of Section 4.2 and its corresponding optimal tableau. Let us analyze the following two changes in the cost vector \mathbf{c} .

First change. Let us suppose that cost vector $\mathbf{c}^T = (3, 2, 1)$ is replaced by $\hat{\mathbf{c}}^T = (4, 3, 1)$. We compute the new reduced cost coefficients $\hat{z}_j - \hat{c}_j$ proceeding like this:

$$\hat{z}_1 - \hat{c}_1 = (4, 3) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 4 = 0.$$

$$\hat{z}_2 - \hat{c}_2 = (4, 3) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 3 = 0.$$

$$\hat{z}_3 - \hat{c}_3 = (4, 3) \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} - 1 = \frac{3}{2}.$$

$$\hat{z}_4 - \hat{c}_4 = (4, 3) \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} - 0 = \frac{1}{2}.$$

$$\hat{z}_5 - \hat{c}_5 = (4, 3) \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} - 0 = 1.$$

Let us update row zero in the tableau to obtain the initial tableau for Model 2.

		x_1	x_2	x_3	x_4	x_5	
		0	0	$\frac{3}{2}$	$\frac{1}{2}$	1	50
4	\mathbf{a}_1	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	5
3	\mathbf{a}_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	10

Since $\hat{z}_j - \hat{c}_j \geq 0$ holds for any j , the tableau is dual feasible, and thus, the basis \mathbf{B} remains optimal. Therefore, the optimal solution remains unchanged: $x_1^* = 5, x_2^* = 10, x_3^* = 0$. The optimal objective value, however, does change: $z^* = 50$. Thus, the sensitivity analysis shows that the firm should still manufacture

5 units of product A and 10 units of B . This is the case because the optimal basis remains unchanged.

Second change. Let us suppose that cost vector $\mathbf{c}^T = (3, 2, 1)$ is replaced by $\hat{\mathbf{c}}^T = (1, 1, 1)$. We compute the new reduced cost coefficients $\hat{z}_j - \hat{c}_j$, and update row zero in the tableau to find the initial tableau for Model 2.

		x_1	x_2	x_3	x_4	x_5	
		0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	15
1	\mathbf{a}_1	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	5
1	\mathbf{a}_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	10

$z_3 - c_3$ is negative; the tableau is not dual feasible, and we have to apply the simplex algorithm to restore feasibility.

		x_1	x_2	x_3	x_4	x_5	
		0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	15
1	\mathbf{a}_1	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	5
1	\mathbf{a}_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	10
		$\frac{1}{2}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{35}{2}$
1	\mathbf{a}_3	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	5
1	\mathbf{a}_2	$\frac{1}{2}$	1	0	$-\frac{1}{4}$	$\frac{3}{4}$	$\frac{25}{2}$

The optimal solution is $x_1^* = 0$, $x_2^* = \frac{25}{2}$, $x_3^* = 5$, and the optimal objective value $z^* = \frac{35}{2}$. In this case, as the benefit obtained from the production of A decreased substantially (from 3 to 1), A has become unprofitable and the firm will stop producing it. Note that product C has become profitable, and thus the resources that were previously used to produce A are now used to produce C . The production level of B is similar before and after the change in the cost vector. \square

4.5 Change in a nonbasic vector \mathbf{a}_j

We focus our sensitivity analysis on the change of a nonbasic vector \mathbf{a}_j . If the column for a basic vector \mathbf{a}_j is changed, \mathbf{B} and \mathbf{B}^{-1} would also change, and

hence, the entire tableau. If this is the case, it is usually recommended to solve the problem from scratch, without performing the sensitivity analysis.

Given a linear model and its optimal tableau, let us suppose that the nonbasic vector \mathbf{a}_j changes to $\hat{\mathbf{a}}_j$. Therefore, we have the original model (Model 1) and a perturbation of it (Model 2).

<u>Model 1</u>	<u>Model 2</u>
$\max z = \mathbf{c}^T \mathbf{x}$	$\max z = \mathbf{c}^T \mathbf{x}$
subject to	subject to
$\mathbf{a}_1 x_1 + \cdots + \mathbf{a}_j x_j + \cdots + \mathbf{a}_n x_n \leq \mathbf{b}$	$\mathbf{a}_1 x_1 + \cdots + \hat{\mathbf{a}}_j x_j + \cdots + \mathbf{a}_n x_n \leq \mathbf{b}$
$x_1, \cdots, x_n \geq 0$	$x_1, \cdots, x_n \geq 0$

The change in the j th column of matrix \mathbf{A} changes column \mathbf{y}_j and the j th reduced cost coefficient $z_j - c_j$ in the tableau (see the shaded column in Figure 4.3). Therefore, we need to update them in the optimal tableau for Model 1 to study the change. The updating may result in a nonoptimal tableau (dual infeasible).

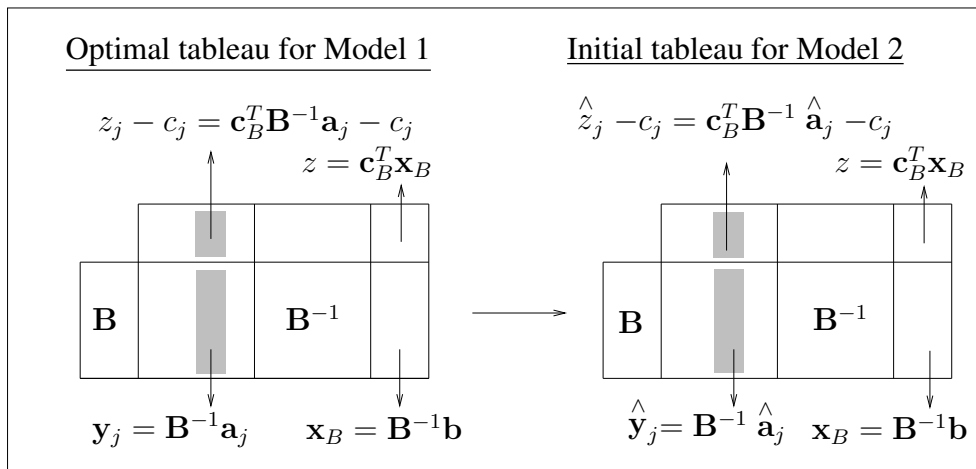


Figure 4.3: Change in vector $\mathbf{a}_j \notin \mathbf{B}$.

As a consequence of the change in vector \mathbf{a}_j , we need to update the following:

$$\hat{\mathbf{y}}_j = \mathbf{B}^{-1} \hat{\mathbf{a}}_j .$$

$$\hat{z}_j - c_j = \mathbf{c}_B^T \mathbf{B}^{-1} \hat{\mathbf{a}}_j - c_j = \mathbf{c}_B^T \hat{\mathbf{y}}_j - c_j.$$

Once $\hat{\mathbf{y}}_j$ and $\hat{z}_j - c_j$ are calculated, they are replaced in the tableau and we find the initial tableau for Model 2. The following two cases can arise:

- **Case 1.** If $\hat{z}_j - c_j \geq 0$, the tableau remains dual feasible, and the solution \mathbf{x}_B and the objective value z remain optimal.
- **Case 2.** If $\hat{z}_j - c_j < 0$, the tableau is no longer optimal since it is not dual feasible. The simplex algorithm will be applied to restore feasibility, starting at the initial tableau for Model 2.

Example. Consider the production problem of Section 4.2 and its corresponding optimal tableau. Let us analyze the following two changes:

First change. The nonbasic vector \mathbf{a}_3 changes like this:

$$\mathbf{a}_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{changes to:} \quad \hat{\mathbf{a}}_3 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Therefore,

$$\hat{\mathbf{y}}_3 = \mathbf{B}^{-1} \hat{\mathbf{a}}_3 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hat{z}_3 - c_3 = (3, 2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1 = 3 - 1 = 2 > 0.$$

Note that the tableau remains dual feasible, and that both the basis and the initial tableau for Model 2 are optimal.

	x_1	x_2	x_3	x_4	x_5	
	0	0	2	$\frac{1}{2}$	$\frac{1}{2}$	35
\mathbf{a}_1	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	5
\mathbf{a}_2	0	1	0	$-\frac{1}{2}$	1	10

The optimal solution is $x_1^* = 5$, $x_2^* = 10$, $x_3^* = 0$ and the optimal objective value is $z^* = 35$. We can say that the change did not affect the production of products A , B and C , and that they provide the same benefit to the firm as before: 35 units.

Second change. Consider the following change:

$$\mathbf{a}_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{changes to:} \quad \hat{\mathbf{a}}_3 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

We compute the updated values,

$$\hat{\mathbf{y}}_3 = \mathbf{B}^{-1} \hat{\mathbf{a}}_3 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ \frac{3}{4} \end{pmatrix}$$

$$\hat{z}_3 - c_3 = (3, 2) \begin{pmatrix} -\frac{1}{4} \\ \frac{3}{4} \end{pmatrix} - 1 = -\frac{1}{4}.$$

The tableau is no longer dual feasible, and we need to use the simplex algorithm to restore the feasibility.

		x_1	x_2	x_3	x_4	x_5			
			0	0	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	35	
3	\mathbf{a}_1	1	0	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$		5	$-\frac{1}{3}$
2	\mathbf{a}_2	0	1	$\frac{3}{4}$	$-\frac{1}{2}$	1		10	
			0	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{5}{6}$	$\frac{115}{3}$	
3	\mathbf{a}_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{1}{6}$		$\frac{25}{3}$	
1	$\hat{\mathbf{a}}_3$	0	$\frac{4}{3}$	1	$-\frac{2}{3}$	$\frac{4}{3}$		$\frac{40}{3}$	

The optimal solution to Model 2 is $x_1^* = \frac{25}{3}$, $x_2^* = 0$ and $x_3^* = \frac{40}{3}$, and the optimal objective value is $z^* = \frac{115}{3}$. In this case, the firm will stop the production of B and start producing C . Note that the resource consumption to produce a unit of product C is very low now (see vector $\hat{\mathbf{a}}_3$) compared with the resource consumption to produce one unit of A or B . Consequently, the production of C has become profitable. \square

4.6 The addition of a new variable

Given a linear model and its optimal tableau, let us suppose that a new variable is added to the model. Therefore, we have the original model (Model 1) and the augmented model (Model 2).

<u>Model 1</u>	<u>Model 2</u>
$\max z = c_1x_1 + \cdots + c_nx_n$	$\max z = c_1x_1 + \cdots + c_nx_n + c_{n+1}x_{n+1}$
subject to	subject to
$\mathbf{a}_1x_1 + \cdots + \mathbf{a}_nx_n \leq \mathbf{b}$	$\mathbf{a}_1x_1 + \cdots + \mathbf{a}_nx_n + \mathbf{a}_{n+1}x_{n+1} \leq \mathbf{b}$
$x_1, \cdots, x_n \geq 0$	$x_1, \cdots, x_n, x_{n+1} \geq 0$

To consider the addition of a new variable x_{n+1} to the model, we need to compute the new column \mathbf{y}_{n+1} and the new reduced cost coefficient $z_{n+1} - c_{n+1}$ in the tableau (see the shaded column in Figure 4.4). We introduce them in the corresponding column of the tableau for Model 1 and obtain the initial tableau for Model 2. The dual feasibility of the initial tableau for the augmented model will depend on the new reduced cost coefficient $z_{n+1} - c_{n+1}$.

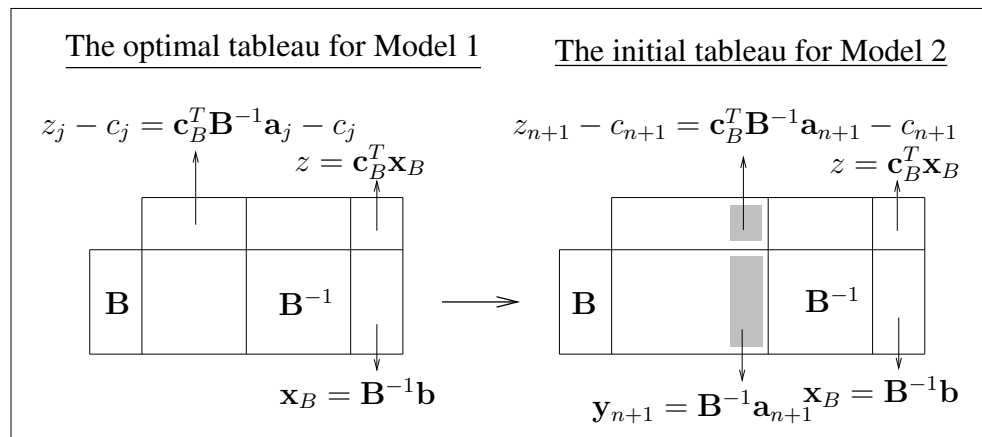


Figure 4.4: Adding a new variable.

So, we compute the new column \mathbf{y}_{n+1} and the new value $z_{n+1} - c_{n+1}$ and obtain the initial tableau for Model 2.

$$\mathbf{y}_{n+1} = \mathbf{B}^{-1} \mathbf{a}_{n+1}.$$

$$z_{n+1} - c_{n+1} = \mathbf{c}_B^T \mathbf{y}_{n+1} - c_{n+1}.$$

The following two cases can arise:

- **Case 1.** If the new $z_{n+1} - c_{n+1} \geq 0$, then the introduction of a new variable x_{n+1} to the model does not affect the optimal solution, since the initial tableau for the augmented model is dual feasible. The solution \mathbf{x}_B and the objective value z remain optimal.
- **Case 2.** If the new $z_{n+1} - c_{n+1} < 0$, then the initial tableau for the augmented model is not optimal since it is not dual feasible. The simplex algorithm will be applied to find the new optimal solution.

Example. Consider the production problem of Section 4.2 and its corresponding optimal tableau. Let us analyze the addition of a new variable and illustrate the two cases just mentioned above.

First case. Let us suppose that the firm wants to consider the production of a new product type D . The new variable x_4 represents the units of the new product being considered for manufacture. The column \mathbf{a}_4 , which represents the consumption of resources needed to produce a unit of the new product, and c_4 , which represents the benefit obtained from each unit produced, are the following:

$$\mathbf{a}_4 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad c_4 = 1.$$

Since the new variable is denoted by x_4 , we rename the two slack variables, and denote them x_5 and x_6 . We compute the new \mathbf{y}_4 and $z_4 - c_4$:

$$\mathbf{y}_4 = \mathbf{B}^{-1} \mathbf{a}_4 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{pmatrix}$$

$$z_4 - c_4 = (3, 2) \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{pmatrix} - 1 = -\frac{3}{2} + \frac{6}{2} - 1 = \frac{1}{2} > 0.$$

Let us update the tableau with the introduction of the new \mathbf{y}_4 and $z_4 - c_4$, to obtain the initial tableau for the augmented model.

		x_1	x_2	x_3	x_4	x_5	x_6	
		0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	35
3	\mathbf{a}_1	1	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	5
2	\mathbf{a}_2	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	1	10

The tableau is dual feasible, and thus, the solution remains optimal: $x_1^* = 5$, $x_2^* = 10$ and $x_3^* = 0$. The objective value remains optimal as well, $z^* = 35$. This means that it is not worth producing the new type of product D , and that the best thing the firm can do is to continue with the production of products A and B .

Second case. Consider the following \mathbf{a}_4 and c_4 for the new product D :

$$\mathbf{a}_4 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad c_4 = 3.$$

We compute the new \mathbf{y}_4 and $z_4 - c_4$:

$$\mathbf{y}_4 = \mathbf{B}^{-1}\mathbf{a}_4 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$z_4 - c_4 = (3, 2) \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} - 3 = \frac{5}{2} - 3 = -\frac{1}{2} < 0.$$

Since $z_4 - c_4$ is negative, the augmented tableau is not dual feasible, and we apply the simplex algorithm.

		x_1	x_2	x_3	x_4	x_5	x_6	
		0	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	35
3	\mathbf{a}_1	1	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	5
2	\mathbf{a}_2	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	10
		1	0	2	0	1	0	40
3	\mathbf{a}_4	2	0	2	1	1	-1	10
2	\mathbf{a}_2	-1	1	$-\frac{3}{2}$	0	-1	$\frac{3}{2}$	5

The optimal solution to the augmented model is: $x_1^* = 0$, $x_2^* = 5$, $x_3^* = 0$ and $x_4^* = 10$, and the new optimal objective value is $z^* = 40$. This means that the manufacture of the new product type D is worthwhile, whilst the production of A should stop. The benefits obtained from the production of B and D are higher than the ones obtained before. \square

4.7 The addition of a new constraint

Given a linear model and its optimal tableau, let us suppose that a new constraint is added to the model. Therefore, we have the original model (Model 1) and the augmented model (Model 2).

<u>Model 1</u>	<u>Model 2</u>
$\max z = c_1x_1 + \cdots + c_nx_n$	$\max z = c_1x_1 + \cdots + c_nx_n$
subject to	subject to
$a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1$	$a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1$
$\vdots \quad \quad \quad \vdots$	$\vdots \quad \quad \quad \vdots$
$a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m$	$a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m$
$x_1, \cdots, x_n \geq 0$	$a_{m+1,1}x_1 + \cdots + a_{m+1,n}x_n \leq b_{m+1}$
	$x_1, \cdots, x_n \geq 0$

Figure 4.5 shows the optimal tableau for Model 1. It illustrates that the columns that correspond to the basic variables contain the identity matrix. However, if a new constraint is added to the tableau just as it is, the resulting tableau will not contain the identity matrix any more.

To remedy this problem, we use elementary operations to express the new constraint in terms of the current basis \mathbf{B} . We replace the last row of the tableau for the augmented model with the appropriate one, which yields a new tableau where all columns are correctly expressed in terms of \mathbf{B} . However, the tableau may be primal infeasible (see Figure 4.6).

With regard to the primal feasibility of the augmented tableau, the following two cases can arise:

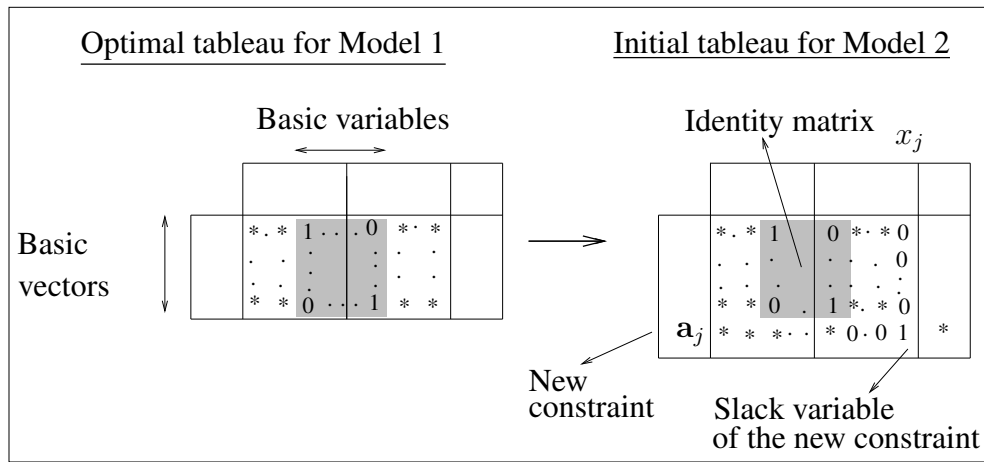


Figure 4.5: Adding a new constraint.

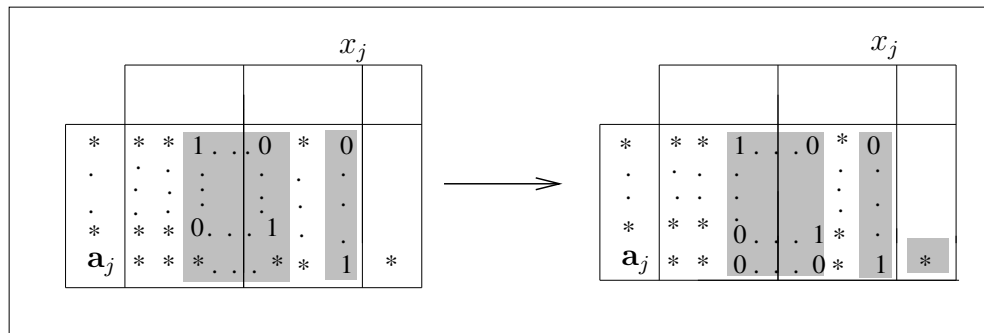


Figure 4.6: A new constraint: using elementary operations to add it

- **Case 1.** If the initial tableau for the augmented model is primal feasible, then the tableau is optimal. The current optimal solution satisfies the new constraint.
- **Case 2.** If the initial tableau for the augmented model is not primal feasible, then the dual simplex algorithm will be applied to find the optimal solution to Model 2.

Example. Consider the production problem of Section 4.2 and its corresponding optimal tableau. Let us analyze the addition of a new constraint, which may be interpreted as if the firm was considering the use of an additional resource in the production of products A , B and C . Our purpose is to illustrate the two cases

previously mentioned.

First case. Let us suppose that 20 units of a new resource are available, and that the firm is considering the following resource consumption in the production of A , B and C :

$$x_1 + x_2 + x_3 \leq 20$$

We add a slack variable to transform the new constraint into an equation.

$$x_1 + x_2 + x_3 + x_6 = 20$$

We add the new equation to the optimal tableau for Model 1.

		x_1	x_2	x_3	x_4	x_5	x_6	
		0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	35
3	\mathbf{a}_1	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	5
2	\mathbf{a}_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	0	10
0	\mathbf{a}_6	1	1	1	0	0	1	20

The columns corresponding to the basic vectors must contain the unitary vectors. However, that is not true in the previous tableau. We use the following elementary operations to write the last row in terms of the new basis $\mathbf{B} = (\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_6)$: row 3 – row 1 – row 2. This leads to the initial tableau for the augmented model.

		x_1	x_2	x_3	x_4	x_5	x_6	
		0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	35
3	\mathbf{a}_1	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	5
2	\mathbf{a}_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	0	10
0	\mathbf{a}_6	0	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	5

The tableau is primal feasible, and thus, the solution and the objective value remain optimal: $x_1^* = 5$, $x_2^* = 10$, $x_3^* = 0$, $z^* = 35$. Note that even though the

optimal solution remains unchanged, and consequently the firm will keep production at the same level as previous to the introduction of the new constraint, the use of the resources has varied; according to the new constraint, each unit of product A consumes one unit of the new resource, just the same as each unit of B .

To see why the solution remains optimal after the new constraint was added to the model, we have to realize that the 20 units of the new resource are enough to continue with the actual production, that is, the solution satisfies the new constraint.

Second case. Let us suppose that a similar constraint to the one used in the previous case is added to the model, but instead of the 20 units available then, now we just have 10:

$$x_1 + x_2 + x_3 \leq 10$$

We add a slack variable to transform the new constraint into an equation.

$$x_1 + x_2 + x_3 + x_6 = 10$$

We add the new equation to the optimal tableau for Model 1.

		x_1	x_2	x_3	x_4	x_5	x_6	
		0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	35
3	\mathbf{a}_1	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	5
2	\mathbf{a}_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	0	10
0	\mathbf{a}_6	1	1	1	0	0	1	10

As in the first case, we need to use elementary operations to write the last row in terms of the new basis $\mathbf{B} = (\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_6)$. We operate like this: row 3 – row 1 – row 2.

		x_1	x_2	x_3	x_4	x_5	x_6	
		0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	35
3	\mathbf{a}_1	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	5
2	\mathbf{a}_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	0	10
0	\mathbf{a}_6	0	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	-5

The initial tableau for the augmented model is not primal feasible. Therefore, the dual simplex algorithm will be used to restore feasibility and find a new optimal solution.

		x_1	x_2	x_3	x_4	x_5	x_6	
		0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	35
3	\mathbf{a}_1	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	5 1
2	\mathbf{a}_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	0	10 -2
0	\mathbf{a}_6	0	0	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	-5
		0	0	$\frac{3}{2}$	$\frac{1}{2}$	0	1	30
3	\mathbf{a}_1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-1	10
2	\mathbf{a}_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	2	0
0	\mathbf{a}_5	0	0	-1	0	1	-2	10

The optimal solution to the augmented model is: $x_1^* = 10$, $x_2^* = 0$ and $x_3^* = 0$, and the optimal objective value is $z^* = 30$. According to this optimal solution, the only profitable product now is A , since the production of B has stopped. Note that the benefit obtained now is lower than the benefit obtained previously.

We should point out that adding a new constraint to a model either reduces the optimal z or leaves it unchanged, since all the constraints of Model 1 remain in Model 2. If the optimal solution to Model 1 does not satisfy the new constraint, then it will not be possible to keep the original benefit; the amount of the new resource available is not enough to maintain the present levels of production. \square

