

Chapter 3

Duality

In the development of the linear programming theory, duality plays an important role both theoretically and in practice. Every linear problem has another linear problem associated, which is called “the dual”. As we will see, when we solve a linear problem, we are simultaneously solving its dual; in fact, the two optimal solutions are present in the optimal simplex tableau.

These are some of the reasons why it is important to take duality into account:

1. The convergence of the simplex algorithm is more sensitive to the number of constraints than to the number of variables of the problem. The fact that the linear problem and its associated dual are solved simultaneously implies that the problem with less constraints can be solved and that the optimal solutions to both can be obtained in a more efficient way.
2. The study of the duality theory provides rich economic interpretations; the solution to the dual provides useful information about the original linear problem.
3. Taking into consideration the duality properties, a new algorithm has been developed to solve linear problems: the dual simplex algorithm. As we will see in future chapters, this new algorithm has important practical applications in sensitivity analysis and integer programming, for instance.

3.1 The dual problem

Definition 3.1.1 (Maximization symmetric form) *A linear problem is said to be written in maximization symmetric form if*

- *The objective is in maximization form*
- *All the constraints are \leq*
- *All variables are nonnegative*

Example. Consider the following linear problem:

$$\begin{aligned} \max z &= x_1 - 3x_2 + x_3 \\ \text{subject to} \\ x_1 + x_2 + x_3 &\geq 2 \\ -x_1 + 2x_2 - x_3 &\leq 3 \\ x_1 - x_2 + 2x_3 &= -1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The problem in maximization symmetric form:

$$\begin{aligned} \max z &= x_1 - 3x_2 + x_3 \\ \text{subject to} \\ -x_1 - x_2 - x_3 &\leq -2 \\ -x_1 + 2x_2 - x_3 &\leq 3 \\ x_1 - x_2 + 2x_3 &\leq -1 \\ -x_1 + x_2 - 2x_3 &\leq 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

□

Definition 3.1.2 (Minimization symmetric form) *A linear problem is said to be written in minimization symmetric form if*

- *The objective is in minimization form*
- *All the constraints are \geq*
- *All variables are nonnegative*

Example. Consider the following linear problem:

$$\begin{aligned} \max z &= x_1 - x_2 \\ \text{subject to} \\ 3x_1 + 2x_2 &\leq 1 \\ x_1 - 2x_2 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The problem in minimization symmetric form:

$$\begin{aligned} \min (-z) &= -x_1 + x_2 \\ \text{subject to} \\ -3x_1 - 2x_2 &\geq -1 \\ x_1 - 2x_2 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

□

3.1.1 Primal-dual relationship

Given a linear problem in maximization symmetric form:

$$\begin{aligned} \max z &= \mathbf{c}^T \mathbf{x} \\ \text{subject to} \\ \mathbf{Ax} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

We refer to it as the *primal* problem. The following problem given in minimization symmetric form is called the corresponding *dual* problem:

$$\begin{aligned} \min G &= \mathbf{b}^T \mathbf{y} \\ \text{subject to} \\ \mathbf{A}^T \mathbf{y} &\geq \mathbf{c} \\ \mathbf{y} &\geq \mathbf{0} \end{aligned}$$

Example. Consider the following linear problem.

$$\begin{aligned} \max z &= 2x_1 - x_2 + 3x_3 \\ \text{subject to} \\ x_1 - x_2 + x_3 &\leq 2 \\ 3x_1 - x_2 + 2x_3 &\leq 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Its corresponding dual problem is the following:

$$\begin{aligned} \min G &= 2y_1 + y_2 \\ \text{subject to} \\ y_1 + 3y_2 &\geq 2 \\ -y_1 - y_2 &\geq -1 \\ y_1 + 2y_2 &\geq 3 \\ y_1, y_2 &\geq 0 \end{aligned}$$

□

3.1.2 Primal-dual correspondence

For any primal problem and the corresponding dual problem, there is a direct correspondence between the parameters of the problems, which may be summarized as follows:

- The size of the constraint matrix \mathbf{A} of the primal problem is $m \times n$; there are m constraints and n variables in it. The constraint matrix of the dual problem is the transpose of the constraint matrix of the primal, \mathbf{A}^T , and therefore, the dual has n constraints and m variables.
- The right hand side vector \mathbf{b} of the primal problem is the vector of cost coefficients of the dual problem.
- The vector of cost coefficients \mathbf{c} of the primal problem is the right hand side vector of the dual problem.
- The number of constraints of the primal problem is equal to the number of variables of the dual problem.
- The number of variables of the primal problem is equal to the number of constraints of the dual problem.

3.1.3 Duality. The general case

In general, many linear problems contain \leq , $=$ or \geq constraints. The variables may also be ≥ 0 , ≤ 0 or unrestricted in sign. In order to compute the dual of any linear problem, we should convert it to the maximization symmetric form and derive from it the dual problem associated. However, in practice it is possible to find immediately the associated dual problem of any given linear problem by using the correspondences of Table 3.1.

Objective function: max	\iff	Objective function: min
i th constraint is $\leq b_i$	\iff	i th variable is ≥ 0
i th constraint is $= b_i$	\iff	i th variable is unrestricted
i th constraint is $\geq b_i$	\iff	i th variable is ≤ 0
i th variable is ≥ 0	\iff	i th constraint is $\geq b_i$
i th variable is unrestricted	\iff	i th constraint is $= b_i$
i th variable is ≤ 0	\iff	i th constraint is $\leq b_i$

Table 3.1: Primal-dual correspondences

In this section we prove some of the correspondences of the table; the rest can be proved likewise.

1st case. If all the constraints in a problem given in maximization form are \geq , then the variables are ≤ 0 in the associated dual problem (third correspondence of the table). Consider the linear problem:

$$\begin{aligned} \max z &= \mathbf{c}^T \mathbf{x} \\ \text{subject to} \\ \mathbf{Ax} &\geq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

The dual of this problem is:

$$\begin{aligned} \min G &= \mathbf{b}^T \mathbf{y} \\ \text{subject to} \\ \mathbf{A}^T \mathbf{y} &\geq \mathbf{c} \\ \mathbf{y} &\leq \mathbf{0} \end{aligned}$$

Proof. We convert the primal problem to the maximization symmetric form by multiplying the set of constraints by -1 .

$$\begin{aligned} \max z &= \mathbf{c}^T \mathbf{x} \\ \text{subject to} \\ -\mathbf{A}\mathbf{x} &\leq -\mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

By using the primal-dual correspondences shown previously (see subsection 3.1.1), we get the dual problem:

$$\begin{aligned} \min G &= -\mathbf{b}^T \mathbf{y} \\ \text{subject to} \\ -\mathbf{A}^T \mathbf{y} &\geq \mathbf{c}^T \\ \mathbf{y} &\geq \mathbf{0} \end{aligned}$$

Substituting $\mathbf{y} = -\mathbf{y}$, the dual problem may be also expressed as follows:

$$\begin{aligned} \min G &= \mathbf{b}^T \mathbf{y} \\ \text{subject to} \\ \mathbf{A}^T \mathbf{y} &\geq \mathbf{c}^T \\ \mathbf{y} &\leq \mathbf{0} \end{aligned}$$

□

2nd case. If all the constraints in a problem given in maximization form are $=$, then the variables are unrestricted in sign in the associated dual problem (second correspondence of the table). Consider the linear problem:

$$\begin{aligned} \max z &= \mathbf{c}^T \mathbf{x} \\ \text{subject to} \\ \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

The dual of this problem is:

$$\begin{aligned} \min G &= \mathbf{b}^T \mathbf{y} \\ \text{subject to} \\ \mathbf{A}^T \mathbf{y} &\geq \mathbf{c} \\ \mathbf{y} &: \text{unrestricted} \end{aligned}$$

Proof. We convert the primal problem to the maximization symmetric form by writing the equality constraint set as two inequalities:

$$\begin{aligned} \max z &= \mathbf{c}^T \mathbf{x} \\ \text{subject to} \\ \mathbf{Ax} &\leq \mathbf{b} \\ -\mathbf{Ax} &\leq -\mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

The dual problem is as follows:

$$\begin{aligned} \min G &= (\mathbf{b}^T, -\mathbf{b}^T) \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \\ \text{subject to} \\ (\mathbf{A}^T, -\mathbf{A}^T) \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} &\geq \mathbf{c} \\ \mathbf{u}, \mathbf{v} &\geq \mathbf{0} \end{aligned}$$

being vectors \mathbf{u} and \mathbf{v} m -dimensional. The dual problem may be written like this:

$$\begin{aligned} \min G &= \mathbf{b}^T (\mathbf{u} - \mathbf{v}) \\ \text{subject to} \\ \mathbf{A}^T (\mathbf{u} - \mathbf{v}) &\geq \mathbf{c} \\ \mathbf{u}, \mathbf{v} &\geq \mathbf{0} \end{aligned}$$

Substituting $\mathbf{y} = \mathbf{u} - \mathbf{v}$, we obtain the dual problem written as follows:

$$\begin{aligned} \min G &= \mathbf{b}^T \mathbf{y} \\ \text{subject to} \\ \mathbf{A}^T \mathbf{y} &\geq \mathbf{c} \\ \mathbf{y} &: \text{unrestricted} \end{aligned}$$

The variables of the dual problem are unrestricted in sign, because they are the subtraction of two positive variables.

Example. Consider the following primal problem.

$$\begin{aligned} \max z &= x_1 - 4x_2 - x_3 \\ \text{subject to} \\ x_1 + x_2 - x_3 &\geq 4 \\ 2x_1 + 3x_2 - 5x_3 &\leq 2 \\ 2x_1 - x_2 + 2x_3 &= 6 \\ x_1 \leq 0, x_2 \geq 0, x_3 &: \text{unrestricted} \end{aligned}$$

The corresponding dual problem is the following (see correspondences of Table 3.1).

$$\begin{aligned} \min G &= 4y_1 + 2y_2 + 6y_3 \\ \text{subject to} \\ y_1 + 2y_2 + 2y_3 &\leq 1 \\ y_1 + 3y_2 - y_3 &\geq -4 \\ -y_1 - 5y_2 + 2y_3 &= -1 \\ y_1 \leq 0, y_2 \geq 0, y_3 &: \text{unrestricted} \end{aligned}$$

□

3.2 Duality proofs

In this section we analyze the main theorems of duality, which establish important relationships between the primal and dual problems and their solutions. All theorems are stated for the primal-dual symmetric forms.

<u>Primal</u>	<u>Dual</u>
$\max z = \mathbf{c}^T \mathbf{x}$	$\min G = \mathbf{b}^T \mathbf{y}$
subject to	subject to
$\mathbf{Ax} \leq \mathbf{b}$	$\mathbf{A}^T \mathbf{y} \geq \mathbf{c}$
$\mathbf{x} \geq \mathbf{0}$	$\mathbf{y} \geq \mathbf{0}$

Theorem 3.2.1 *The dual of the dual is the primal.*

Proof. Consider the following linear problem:

$$\begin{aligned} \min G &= \mathbf{b}^T \mathbf{y} \\ \text{subject to} \\ \mathbf{A}^T \mathbf{y} &\geq \mathbf{c} \\ \mathbf{y} &\geq \mathbf{0} \end{aligned}$$

We rewrite the problem in maximization symmetric form,

$$\begin{aligned} -\max (-G) &= -\mathbf{b}^T \mathbf{y} \\ \text{subject to} \\ -\mathbf{A}^T \mathbf{y} &\leq -\mathbf{c} \\ \mathbf{y} &\geq \mathbf{0} \end{aligned}$$

and compute its dual, according to the primal-dual correspondence:

$$\begin{aligned} -\min (-z) &= -\mathbf{c}^T \mathbf{x} \\ \text{subject to} \\ -\mathbf{Ax} &\geq -\mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

Transforming the objective function and the set of constraints, we get the primal problem.

$$\begin{aligned} \max z &= \mathbf{c}^T \mathbf{x} \\ \text{subject to} \\ \mathbf{Ax} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

□

We can conclude from the previous theorem that, if the primal objective function is minimized, then the dual objective function is maximized. Therefore, correspondences in Table 3.1 are read from right to left when calculating the dual of a minimization problem.

Example. Consider the following primal problem:

$$\begin{aligned} \min z &= x_1 - 4x_2 - x_3 \\ \text{subject to} \\ x_1 + x_2 - x_3 &\geq 4 \\ 2x_1 + 3x_2 - 5x_3 &\leq 2 \\ 2x_1 - x_2 + 2x_3 &= 6 \\ x_1 \leq 0, x_2 \geq 0, x_3 &: \text{unrestricted} \end{aligned}$$

By using the correspondences of Table 3.1 we get the associated dual problem.

$$\begin{aligned} \max G &= 4y_1 + 2y_2 + 6y_3 \\ \text{subject to} \\ y_1 + 2y_2 + 2y_3 &\geq 1 \\ y_1 + 3y_2 - y_3 &\leq -4 \\ -y_1 - 5y_2 + 2y_3 &= -1 \\ y_1 \geq 0, y_2 \leq 0, y_3 &: \text{unrestricted} \end{aligned}$$

□

Theorem 3.2.2 (Weak duality) *Let \mathbf{x} and \mathbf{y} be any feasible solutions to the primal and dual problems, respectively. The following inequality holds:*

$$z = \mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y} = G.$$

Proof.

Since \mathbf{x} is a feasible solution to the primal problem, $\mathbf{Ax} \leq \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$ are satisfied. Since \mathbf{y} is a feasible solution to the dual problem, $\mathbf{A}^T \mathbf{y} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$ are satisfied.

Multiplying $\mathbf{Ax} \leq \mathbf{b}$ on the left by \mathbf{y}^T , and $\mathbf{A}^T \mathbf{y} \geq \mathbf{c}$ by \mathbf{x}^T , we get:

$$\mathbf{y}^T \mathbf{Ax} \leq \mathbf{y}^T \mathbf{b} = \mathbf{b}^T \mathbf{y}.$$

$$\mathbf{x}^T \mathbf{A}^T \mathbf{y} \geq \mathbf{x}^T \mathbf{c} = \mathbf{c}^T \mathbf{x}.$$

Since $\mathbf{x}^T \mathbf{A}^T \mathbf{y} = \mathbf{y}^T \mathbf{A} \mathbf{x}$, the following holds:

$$z = \mathbf{c}^T \mathbf{x} \leq \mathbf{y}^T \mathbf{A} \mathbf{x} \leq \mathbf{b}^T \mathbf{y} = G.$$

□

The weak duality theorem indicates that the maximum objective value for the primal problem is a lower bound on the minimum objective value of the dual problem. Similarly, the minimum objective value for the dual problem is an upper bound on the maximum objective value of the primal problem. The following corollaries are immediate consequences of the previous theorem.

Corollary 3.2.1 *If \mathbf{x}^* and \mathbf{y}^* are feasible solutions to the primal and dual problem such that $\mathbf{c}^T \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*$ holds, then \mathbf{x}^* and \mathbf{y}^* are optimal solutions to the primal and dual problems respectively.*

Proof. The weak duality theorem states that, for any feasible solutions \mathbf{x} and \mathbf{y} to the primal and dual problem, the following holds:

$$\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}.$$

Since \mathbf{y}^* is a solution to the dual problem, $\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}^*$ holds. $\mathbf{c}^T \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*$ is verified, and therefore, for any solution \mathbf{x} to the primal problem the following holds:

$$\mathbf{c}^T \mathbf{x} \leq \mathbf{c}^T \mathbf{x}^*.$$

We conclude that \mathbf{x}^* is an optimal solution to the primal problem.

Similarly, since $\mathbf{b}^T \mathbf{y}^* = \mathbf{c}^T \mathbf{x}^* \leq \mathbf{b}^T \mathbf{y}$, for any solution \mathbf{y} to the dual problem the following holds:

$$\mathbf{b}^T \mathbf{y}^* \leq \mathbf{b}^T \mathbf{y}.$$

We conclude that \mathbf{y}^* is an optimal solution to the dual problem.

□

Corollary 3.2.2 *If the primal problem is feasible and unbounded, the dual problem is infeasible.*

Proof. Taking into account that $\mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}$ holds for any feasible solutions \mathbf{x} and \mathbf{y} , if the primal objective value is unbounded, then there does not exist any solution \mathbf{y} to the dual problem that is an upper bound of the primal problem. □

The following corollary is proved in a similar way.

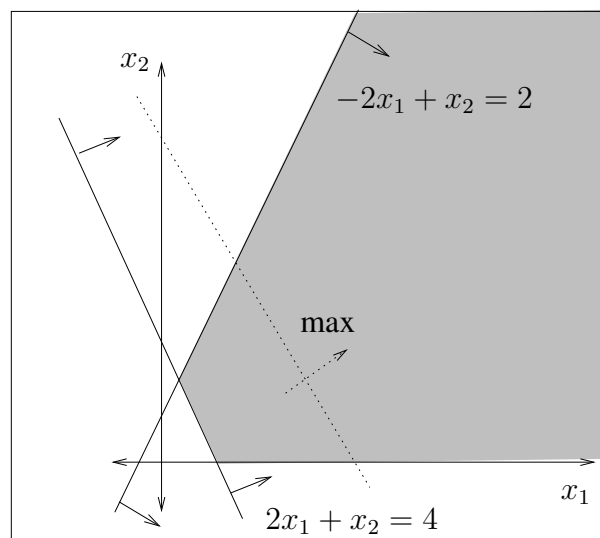
Corollary 3.2.3 *If the dual problem is feasible and unbounded, the primal problem is infeasible.*

If the primal problem is infeasible, the dual problem may be either infeasible or unbounded. Similarly, if the dual problem is infeasible, the primal problem may be either infeasible or unbounded.

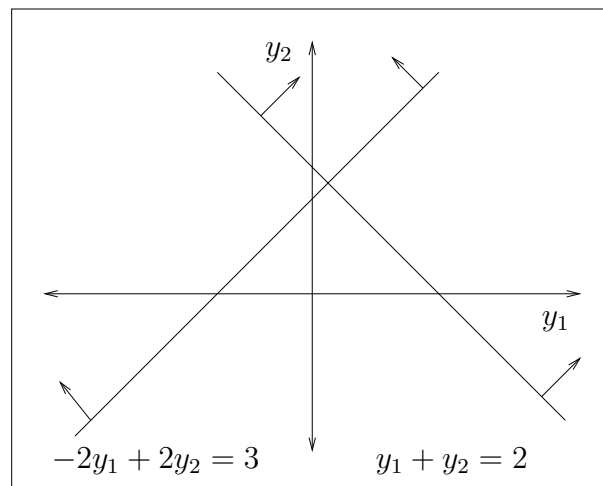
Example. Let us check Corollary 3.2.2 for the following primal and dual problems. In this case, the primal is unbounded and the dual infeasible.

$$\begin{array}{ll}
 \max z = 3x_1 + 2x_2 & \min G = 2y_1 + 4y_2 \\
 \text{subject to} & \text{subject to} \\
 -2x_1 + x_2 \leq 2 & -2y_1 + 2y_2 \geq 3 \\
 2x_1 + x_2 \geq 4 & y_1 + y_2 \geq 2 \\
 x_1, x_2 \geq 0 & y_1 \geq 0, y_2 \leq 0
 \end{array}$$

The graphical solution shows that the primal problem is unbounded.



The feasibility region for the dual problem is empty; that is, the dual problem is infeasible.



□

Theorem 3.2.3 (The fundamental principle of duality) *If an optimal solution \mathbf{x}^* to the primal problem exists, then an optimal solution \mathbf{y}^* to the dual problem exists. Similarly, if an optimal solution \mathbf{y}^* to the dual problem exists, then an optimal solution \mathbf{x}^* to the primal problem exists. In both cases*

$$z^* = \mathbf{c}^T \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^* = G^*$$

holds.

Example. Consider the following primal and dual problems:

$$\max z = 2x_1 + 3x_2$$

subject to

$$x_1 + x_2 \leq 2$$

$$2x_1 - x_2 \leq 3$$

$$x_1 + 3x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

$$\min G = 2y_1 + 3y_2 + 5y_3$$

subject to

$$y_1 + 2y_2 + y_3 \geq 2$$

$$y_1 - y_2 + 3y_3 \geq 3$$

$$y_1, y_2, y_3 \geq 0$$

For any primal and dual feasible solutions $z \leq G$ holds. For instance, we may verify that $\mathbf{x}^T = (1, 1)$ is a primal feasible solution (it satisfies the primal constraints), and $\mathbf{y}^T = (1, 1, 1)$ is a dual feasible solution; by substituting the

solutions into the respective objective functions, we may verify that $z = 5 \leq 10 = G$ holds, as the weak duality theorem predicts.

The optimal primal and dual solutions are the following:

$$\mathbf{x}^{*T} = \left(\frac{1}{2}, \frac{3}{2}\right), \quad \mathbf{y}^{*T} = \left(\frac{3}{2}, 0, \frac{1}{2}\right).$$

The objective value for both problems is the same, $z^* = \frac{11}{2} = G^*$. \square

3.3 The principle of complementary slackness

Applying the principle of complementary slackness, we can use the primal optimal solution to solve the corresponding dual problem, and viceversa. The conditions are stated in the following theorem.

Theorem 3.3.1 (Complementary slackness) *Any feasible solutions to the primal and dual problems \mathbf{x}^* and \mathbf{y}^* are optimal if and only if the following holds:*

$$\mathbf{x}^{*T}(\mathbf{A}^T \mathbf{y}^* - \mathbf{c}) + \mathbf{y}^{*T}(\mathbf{b} - \mathbf{A}\mathbf{x}^*) = 0.$$

From the interpretation of this theorem we obtain the complementary slackness conditions. Remember that the primal and dual problems are given in symmetric form.

3.3.1 Interpretation of the complementary slackness conditions

We may write the primal and dual constraints in the following way, being \mathbf{x}^* and \mathbf{y}^* optimal solutions to the respective problems:

$$\begin{aligned} \mathbf{A}^T \mathbf{y}^* - \mathbf{c} &\geq 0. \\ \mathbf{b} - \mathbf{A}\mathbf{x}^* &\geq 0. \end{aligned}$$

Since the primal and dual optimal solutions \mathbf{x}^* and \mathbf{y}^* are positive, by multiplying the previous inequalities on the left by \mathbf{x}^{*T} and \mathbf{y}^{*T} , respectively, we get:

$$\begin{aligned} \mathbf{x}^{*T}(\mathbf{A}^T \mathbf{y}^* - \mathbf{c}) &\geq 0. \\ \mathbf{y}^{*T}(\mathbf{b} - \mathbf{A}\mathbf{x}^*) &\geq 0. \end{aligned}$$

The complementary slackness theorem states that the sum of the left hand side of the two previous inequalities is zero. Taking into account that the two addends are greater than or equal to zero, we conclude that each of them must be zero, that is,

$$\begin{aligned}\mathbf{x}^{*T}(\mathbf{A}^T\mathbf{y}^* - \mathbf{c}) &= 0. \\ \mathbf{y}^{*T}(\mathbf{b} - \mathbf{A}\mathbf{x}^*) &= 0.\end{aligned}$$

According to the foregoing equations, the product of two nonnegative factors is zero; this implies that if one of them is not zero, then the other must be. Thus, we obtain the following conclusions, which enable us to compute the optimal solution to one of the problems from the optimal solution to the other.

1. If a primal variable is positive, that is to say, greater than zero, then the corresponding constraint in the dual must hold with equality.

$$\mathbf{x}^* > \mathbf{0} \Rightarrow \mathbf{A}^T\mathbf{y}^* - \mathbf{c} = \mathbf{0}.$$

2. If a constraint of the primal is not binding, that is to say, if the equality does not hold, then the corresponding dual variable must be zero.

$$\mathbf{A}\mathbf{x}^* < \mathbf{b} \Rightarrow \mathbf{y}^* = \mathbf{0}.$$

3. If a dual variable is positive, that is to say, greater than zero, then the corresponding constraint in the primal must hold with equality.

$$\mathbf{y}^* > \mathbf{0} \Rightarrow \mathbf{A}\mathbf{x}^* - \mathbf{b} = \mathbf{0}.$$

4. If a constraint of the dual is not binding, that is to say, if the equality does not hold, then the corresponding primal variable must be zero.

$$\mathbf{A}^T\mathbf{y}^* > \mathbf{c} \Rightarrow \mathbf{x}^* = \mathbf{0}.$$

Example. Consider the following linear problem:

$$\begin{aligned}\max z &= 3x_1 + x_2 - 2x_3 \\ \text{subject to} \\ x_1 + 2x_2 + x_3 &\leq 5 \\ 2x_1 - x_2 + 3x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0\end{aligned}$$

The optimal solution to the problem is $\mathbf{x}^{*T} = (\frac{13}{5}, \frac{6}{5}, 0)$. The dual problem is the following:

$$\begin{aligned} \min G &= 5y_1 + 4y_2 \\ \text{subject to} \\ y_1 + 2y_2 &\geq 3 \\ 2y_1 - y_2 &\geq 1 \\ y_1 + 3y_2 &\geq -2 \\ y_1, y_2 &\geq 0 \end{aligned}$$

We introduce three slack variables to the problem and get:

$$\begin{aligned} \min G &= 5y_1 + 4y_2 + 0y_3 + 0y_4 + 0y_5 \\ \text{subject to} \\ y_1 + 2y_2 - y_3 &= 3 \\ 2y_1 - y_2 - y_4 &= 1 \\ y_1 + 3y_2 - y_5 &= -2 \\ y_1, y_2, y_3, y_4, y_5 &\geq 0 \end{aligned}$$

Using the interpretation of the theorem of complementary slackness, we compute the optimal solution to the dual problem, proceeding like this:

<u>Primal variables</u>	<u>Dual constraints</u>
$x_1^* = \frac{13}{5} > 0$	$\Rightarrow y_1^* + 2y_2^* = 3 \Rightarrow y_3^* = 0$
$x_2^* = \frac{6}{5} > 0$	$\Rightarrow 2y_1^* - y_2^* = 1 \Rightarrow y_4^* = 0$
$x_3^* = 0$	$\Rightarrow y_1^* + 3y_2^* - y_5^* = -2$
<u>Primal constraints</u>	<u>Dual variables</u>
$\frac{13}{5} + 2 \times \frac{6}{5} = 5$	\Rightarrow we compute y_1^*
$2 \times \frac{13}{5} - \frac{6}{5} = 4$	\Rightarrow we compute y_2^*

We substitute the values of y_3^* and y_4^* in the dual constraint set, and we have:

$$\begin{aligned} y_1^* + 2y_2^* &= 3 \\ 2y_1^* - y_2^* &= 1 \\ y_1^* + 3y_2^* - y_5^* &= -2 \end{aligned}$$

By solving the system of linear equations, we obtain the optimal solution to the dual problem.

$$y_1^* = 1, y_2^* = 1, y_3^* = 0, y_4^* = 0, y_5^* = 6.$$

□

3.4 The optimal solution to the dual problem

Theorem 3.4.1 Consider the symmetric form of duality, and let \mathbf{B} be an optimal basis for the primal problem. Then, $\mathbf{y}^{*T} = \mathbf{c}_B^T \mathbf{B}^{-1}$ is an optimal solution to the dual problem.

Proof. We consider the constraints of the primal problem in maximization symmetric form, and add vector \mathbf{x}_s of slack variables, which yields:

$$\begin{aligned} \mathbf{A}\mathbf{x} + \mathbf{I}\mathbf{x}_s &= \mathbf{b} \\ \mathbf{x}, \mathbf{x}_s &\geq \mathbf{0} \end{aligned}$$

If \mathbf{B} is an optimal basis for the primal problem, and \mathbf{x}_B an optimal basic feasible solution, then $z_j - c_j \geq 0$ holds for each vector \mathbf{a}_j of matrix \mathbf{A} .

$$z_j = \mathbf{c}_B^T \mathbf{y}_j = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{a}_j.$$

Since \mathbf{B} is optimal, $z_j \geq c_j$ holds for each vector \mathbf{a}_j of matrix \mathbf{A} , that is,

$$\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A} \geq \mathbf{c}^T.$$

The foregoing inequality can be equally written as follows:

$$\mathbf{A}^T (\mathbf{c}_B^T \mathbf{B}^{-1})^T \geq \mathbf{c},$$

which is the dual constraint set. Thus, $\mathbf{y}^* = (\mathbf{c}_B^T \mathbf{B}^{-1})^T$ is a solution to the dual problem.

In order to check the feasibility of the solution, we compute the $z_j - c_j$ corresponding to the vectors of submatrix \mathbf{I} .

$$\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{I} \geq \mathbf{c}_I^T.$$

Since vector \mathbf{x}_s corresponds to slack variables, $\mathbf{c}_I = \mathbf{0}$ holds and thus:

$$\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{I} = \mathbf{c}_B^T \mathbf{B}^{-1} \geq \mathbf{0}^T.$$

Therefore, all the components of vector $\mathbf{y}^* = (\mathbf{c}_B^T \mathbf{B}^{-1})^T$ are nonnegative.

We next show that the solution \mathbf{y}^* is an optimal solution to the dual problem. We prove it by checking that the primal problem and the dual problem have the same objective value.

$$z^* = \mathbf{c}_B^T \mathbf{x}_B = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b} = \mathbf{b}^T (\mathbf{c}_B^T \mathbf{B}^{-1})^T = \mathbf{b}^T \mathbf{y}^* = G^*.$$

Therefore $\mathbf{y}^* = (\mathbf{c}_B^T \mathbf{B}^{-1})^T$ is an optimal solution to the dual problem. \square

3.4.1 A solution to the dual problem in the tableau

We turn now to see that once the primal problem is solved, an optimal solution to the dual problem is readily obtainable in the optimal tableau.

From Theorem 3.4.1 we know that if \mathbf{B} be is an optimal basis for the primal problem, then $\mathbf{y}^{*T} = \mathbf{c}_B^T \mathbf{B}^{-1}$ is an optimal solution to the dual problem. We now show that such vector appears in the optimal tableau at row zero (row of reduced cost coefficients $z_j - c_j$), where the identity matrix \mathbf{I} appeared initially.

$$z_j - c_j = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{a}_j - c_j.$$

If we compute all the $z_j - c_j$ of the corresponding vectors in this identity matrix, we get the following vector:

$$\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{I} - \mathbf{c}_I^T = \mathbf{c}_B^T \mathbf{B}^{-1} - \mathbf{c}_I^T.$$

Therefore, since $\mathbf{y}^{*T} = \mathbf{c}_B^T \mathbf{B}^{-1}$ is the optimal solution to the dual problem, we need to add \mathbf{c}_I^T to the reduced cost coefficients $z_j - c_j$ to obtain it, where \mathbf{c}_I is the m -vector representing the cost coefficients of the variables corresponding to the columns of the original identity matrix. Two cases may be distinguished:

- If the columns of matrix \mathbf{I} correspond to slack variables, then $\mathbf{c}_I = \mathbf{0}$ holds.
- If there are artificial variables in the columns corresponding to the initial matrix \mathbf{I} , then the cost coefficients of the artificial variables are M , and they appear in vector \mathbf{c}_I , due to the penalty of the objective function.

Example. Consider the problem of page 101.

$$\max z = 3x_1 + x_2 - 2x_3$$

subject to

$$x_1 + 2x_2 + x_3 \leq 5$$

$$2x_1 - x_2 + 3x_3 \leq 4$$

$$x_1, x_2, x_3 \geq 0$$

We add two slack variables to the problem and construct the initial tableau.

	x_1	x_2	x_3	x_4	x_5	
	-3	-1	2	0	0	0
\mathbf{a}_4	1	2	1	1	0	5
\mathbf{a}_5	2	-1	3	0	1	4

By applying the simplex algorithm, we obtain the following optimal tableau:

	x_1	x_2	x_3	x_4	x_5	
	0	0	6	1	1	9
\mathbf{a}_2	0	1	$-\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{6}{5}$
\mathbf{a}_1	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{13}{5}$

The optimal solution to the primal problem is:

$$x_1^* = \frac{13}{5}, x_2^* = \frac{6}{5}, x_3^* = 0, z^* = 9.$$

Let us extract the optimal solution to the dual problem directly from the optimal tableau computed for the primal problem. The initial identity matrix, $\mathbf{B} = \mathbf{I} = (\mathbf{a}_4, \mathbf{a}_5)$, corresponds to the two slack variables. In the optimal tableau, \mathbf{B}^{-1} appears at the columns corresponding to the variables x_4 and x_5 , and the optimal solution to the dual problem is obtained from the same columns in row zero.

$$(z_4 - c_4, z_5 - c_5) = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{I} - \mathbf{c}_I^T = (1, 1) - \mathbf{c}_I^T.$$

Since $\mathbf{c}_I^T = (c_4, c_5) = (0, 0)$, the optimal solution to the dual problem is :

$$\mathbf{y}^{*T} = \mathbf{c}_B^T \mathbf{B}^{-1} = (1, 1).$$

The optimal objective value of the dual problem is $G^* = 9$. □

Example. Consider the following linear problem:

$$\begin{aligned} \min z &= x_1 + 2x_2 \\ \text{subject to} \\ 4x_1 + 3x_2 &\leq 12 \\ x_1 + 3x_2 &\geq 6 \\ 2x_1 + x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

To solve the problem with the simplex algorithm, besides adding slack variables, we need to add two artificial variables, w_1 and w_2 .

$$\max(-z) = -x_1 - 2x_2 + 0x_3 + 0x_4 + 0x_5 - Mw_1 - Mw_2$$

subject to

$$4x_1 + 3x_2 + x_3 = 12$$

$$x_1 + 3x_2 - x_4 + w_1 = 6$$

$$2x_1 + x_2 - x_5 + w_2 = 4$$

$$x_1, x_2, x_3, x_4, x_5, w_1, w_2 \geq 0$$

The initial canonical basis is: $\mathbf{B} = \mathbf{I} = (\mathbf{a}_3, \mathbf{a}_{w1}, \mathbf{a}_{w2})$.

		x_1	x_2	x_3	x_4	x_5	w_1	w_2		
		$-3M + 1$	$-4M + 2$	0	M	M	0	0	$-10M$	
0	\mathbf{a}_3	4	3	1	0	0	0	0	12	1
$-M$	\mathbf{a}_{w1}	1	3	0	-1	0	1	0	6	
$-M$	\mathbf{a}_{w2}	2	1	0	0	-1	0	1	4	$\frac{1}{3}$
		$-\frac{5}{3}M + \frac{1}{3}$	0	$0 - \frac{1}{3}M + \frac{2}{3}$	M	$\frac{4}{3}M - \frac{2}{3}$	0	0	$-2M - 4$	
0	\mathbf{a}_3	3	0	1	1	0	-1	0	6	$\frac{9}{5}$
-2	\mathbf{a}_2	$\frac{1}{3}$	1	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	2	$\frac{1}{5}$
$-M$	\mathbf{a}_{w2}	$\frac{5}{3}$	0	0	$\frac{1}{3}$	-1	$-\frac{1}{3}$	1	2	
		0	0	0	$\frac{3}{5}$	$\frac{1}{5}$	$M - \frac{3}{5}$	$M - \frac{1}{5}$	$-\frac{22}{5}$	
0	\mathbf{a}_3	0	0	1	$\frac{2}{5}$	$\frac{9}{5}$	$-\frac{2}{5}$	$-\frac{9}{5}$	$\frac{12}{5}$	
-2	\mathbf{a}_2	0	1	0	$-\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{8}{5}$	
-1	\mathbf{a}_1	1	0	0	$\frac{1}{5}$	$-\frac{3}{5}$	$-\frac{1}{5}$	$\frac{3}{5}$	$\frac{6}{5}$	

The optimal solution to the primal problem is:

$$x_1^* = \frac{6}{5}, \quad x_2^* = \frac{8}{5}, \quad -z^* = -\frac{22}{5} \quad \Rightarrow \quad z^* = \frac{22}{5}.$$

Let us obtain the optimal solution to the dual problem from the optimal simplex tableau. The identity matrix in the first tableau is in the columns corresponding to the variables x_3 , w_1 and w_2 . In the same columns of the optimal tableau we find \mathbf{B}^{-1} , as well as the optimal solution to the dual problem in row zero.

$$(z_3 - c_3, z_{w_1} - c_{w_1}, z_{w_2} - c_{w_2}) = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{I} - \mathbf{c}_I^T = (0, M - \frac{3}{5}, M - \frac{1}{5}).$$

Since $\mathbf{c}_I^T = (c_3, c_{w_1}, c_{w_2}) = (0, -M, -M)$, by adding these cost coefficients we obtain:

$$\mathbf{y}^{*T} = \mathbf{c}_B^T \mathbf{B}^{-1} - \mathbf{c}_I^T + \mathbf{c}_I^T = (0, M - \frac{3}{5}, M - \frac{1}{5}) + (0, -M, -M).$$

We would say that the optimal solution to the dual problem is:

$$\mathbf{y}^{*T} = (0, -\frac{3}{5}, -\frac{1}{5}).$$

However, the sign of the variables has to be verified. If we write the dual problem, we will see that variables y_2 and y_3 are nonnegative. In this case, the values obtained from the optimal tableau are negative. The reason for this is that the transformation applied to the objective function of the primal problem, before using the simplex algorithm, has an effect in the reduced cost coefficients $z_j - c_j$ of row zero. In this example, the original objective function is in minimization form, and we transform it to the maximization form; that is why the sign of the variables is not correct. Hence, the optimal solution to the dual problem is:

$$y_1^* = 0, \quad y_2^* = \frac{3}{5}, \quad y_3^* = \frac{1}{5}, \quad G^* = \frac{22}{5}.$$

□

If we transform a constraint in a linear model by multiplying it by -1 before applying the simplex algorithm, the sign of the value in row zero that will be assigned to the corresponding dual variable will be affected. This change of sign may be explained analyzing the primal-dual correspondences of Table 3.1 on page 91. We can state that the absolute value of the components of vector $\mathbf{c}_B^T \mathbf{B}^{-1}$ are equal to the absolute value of the optimal values of the dual variables.

3.5 Economic interpretation of duality

The optimal solution to a linear model determines the optimal resource allocation to maximize revenue, when resources are limited. In this section we will see

that the optimal solution to the dual model gives us useful information about the convenience of changing the amount of resources available (the right-hand-side of the constraints). We now introduce the concept of shadow price, which helps to make interesting economic interpretations.

3.5.1 Shadow prices

Consider a linear model and its optimal basis \mathbf{B} . Let \mathbf{x}^* be the optimal solution to that primal problem, and z^* its corresponding optimal objective value. Similarly, let \mathbf{y}^* be the optimal solution to the dual problem, and G^* its corresponding optimal objective value. Note that \mathbf{x}^* , z^* , \mathbf{y}^* and G^* are computed in terms of the optimal basis \mathbf{B} .

Let us assume that the right-hand-side vector \mathbf{b} changes to $\mathbf{b} + \Delta\mathbf{b}$. We will analyze the effect the change has on all the calculations in the tableau corresponding to the optimal basis \mathbf{B} , assuming that the current basis remains optimal (the current basic feasible solution remains feasible).

- The new basic feasible solution is:

$$\hat{\mathbf{x}}_B = \mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) = \mathbf{x}_B + \mathbf{B}^{-1}\Delta\mathbf{b}.$$

- The change in the right-hand-side vector does not have any effect on the values $z_j - c_j$ of row zero.

$$z_j - c_j = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{a}_j - c_j.$$

- The objective values of the primal and dual problems change, according to the variations on vector \mathbf{b} (that is, the increase/decrease specified by $\Delta\mathbf{b}$).

$$\hat{G}^* = \mathbf{y}^{*T}(\mathbf{b} + \Delta\mathbf{b}) = \mathbf{y}^{*T}\mathbf{b} + \mathbf{y}^{*T}\Delta\mathbf{b} = G^* + \mathbf{y}^{*T}\Delta\mathbf{b} = z^* + \mathbf{y}^{*T}\Delta\mathbf{b}.$$

Consequently, if a change in vector \mathbf{b} does not violate optimality, that is, if the new basic solution $\hat{\mathbf{x}}_B$ remains feasible, the solution will be affected in the following way:

- The optimal solution to the dual problem remains the same.
- The optimal solution to the primal problem changes. The variation is specified by $\mathbf{B}^{-1}\Delta\mathbf{b}$.

- The primal and dual objective values increase/decrease in $\mathbf{y}^{*T} \Delta \mathbf{b}$.

In summary, if the right-hand-side vector changes and $\hat{\mathbf{x}}_B = \mathbf{B}^{-1}(\mathbf{b} + \Delta \mathbf{b}) \geq 0$, $\hat{\mathbf{x}}_B$ is the optimal solution to the primal problem and \mathbf{y}^* remains optimal to the dual problem. The optimal objective value for both problems increases $\mathbf{y}^{*T} \Delta \mathbf{b}$ over its original optimal value.

In order to interpret the meaning of each of the dual variables, we assume that $\Delta b_i = 1$ and all the rest are zero. Thus, the objective value is increased:

$$\mathbf{y}^{*T} \Delta \mathbf{b} = (y_1^*, \dots, y_i^*, \dots, y_m^*) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = y_i^*.$$

Thus, if the i th right-hand-side (i th resource) is increased in one unit, $\Delta b_i = 1$, and all the rest of the resources remain unchanged, then the optimal value of the dual variable y_i^* is the rate of change (increase) of the optimal objective value. The following definition states when the optimal dual variables are considered to be shadow prices.

Definition 3.5.1 (Shadow price) *The optimal dual variable y_i^* is said to be the shadow price of the i th right-hand-side value, $i = 1, \dots, m$, if increasing/decreasing a unit in the i th right-hand-side value, and keeping the rest of the right-hand-side values unchanged, does not violate the optimality of the tableau.*

Therefore, the shadow price of a resource tells us the increase/decrease in profit resulting from a unit increase/decrease in the availability of the resource. In other words, shadow prices enable us to make economic interpretations such as: how to increase/decrease future profits by increasing/decreasing the availability of the resources.

Example. Consider the example on page 104. Let us check whether the optimal dual variables $y_1^* = 1$ and $y_2^* = 1$ are shadow prices of resources b_1 and b_2 .

- Suppose we change $b_1 = 5$ to $b_1 + \Delta b_1 = 6$, that is, $\Delta b_1 = 1$,

$$\hat{\mathbf{x}}_B = \mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{8}{5} \\ \frac{14}{5} \end{pmatrix}$$

The optimality of the tableau is not violated, since the basic solution corresponding to the new right-hand-side vector remains feasible, $\hat{\mathbf{x}}_B \geq \mathbf{0}$. Hence, we can state that y_1^* is the shadow price of resource b_1 . The new optimal objective value increases as follows:

$$\hat{z}^* = z^* + y_1^* = 9 + 1 = 10.$$

- Suppose we change $b_2 = 4$ to $b_2 + \Delta b_2 = 3$, that is, $\Delta b_2 = -1$.

$$\hat{\mathbf{x}}_B = \mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{7}{5} \\ \frac{11}{5} \end{pmatrix}$$

The optimality of the tableau is not violated, since there is primal feasibility in the tableau, $\hat{\mathbf{x}}_B \geq \mathbf{0}$. Hence, we can state that y_2^* is the shadow price of resource b_2 . The new optimal objective value decreases as follows:

$$\hat{z}^* = z^* - y_2^* = 9 - 1 = 8.$$

In this case, the objective value decreases, since the unit change applied to the resource was negative.

□

Example. Consider the example on page 105. The optimal solution to the dual problem is:

$$y_1^* = 0, \quad y_2^* = \frac{3}{5}, \quad y_3^* = \frac{1}{5}.$$

Let us analyze the effect of a unit increase in the first right-hand-side value of the model in the example, that is, we change $b_1 = 12$ to $b_1 + \Delta b_1 = 13$.

$$\hat{\mathbf{x}}_B = \mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) = \begin{pmatrix} 1 & -\frac{2}{5} & -\frac{9}{5} \\ 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & -\frac{1}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 13 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{17}{5} \\ \frac{8}{5} \\ \frac{6}{5} \end{pmatrix} \geq \mathbf{0}.$$

Since there is primal feasibility in the tableau, $\hat{\mathbf{x}}_B \geq \mathbf{0}$, the optimality of the tableau is not violated. Hence, we can state that y_1^* is the shadow price of the first right-hand-side value. However, the optimal objective value does not vary because the shadow price is $y_1^* = 0$.

$$\hat{z}^* = z^* + y_1^* = \frac{22}{5} + 0 = \frac{22}{5}.$$

The fact that the first constraint of the model has a shadow-price of zero means that increasing the available amount of the resource will not increase the profit, which is logical because we are not currently using all of the resource. We can easily verify it by substituting the optimal solution in the first constraint. Remember that $x_1^* = \frac{6}{5}$, $x_2^* = \frac{8}{5}$ and that the first constraint is $4x_1 + 3x_2 \leq 12$.

$$4 \times \frac{6}{5} + 3 \times \frac{8}{5} < 12.$$

Thus, no improvement is achieved from increasing the amount of the first resource. Instead, we should analyze the convenience of reducing the amount of it. \square

3.5.2 The economic cost of the primal variables and the interpretation of the simplex method

The interpretation of the dual problem also provides an economic interpretation of what the simplex method does in the primal problem. In this subsection, we define the economic cost of the primal variables and give an economic interpretation of the simplex method. To better understand it, we consider the following problem.

A firm produces four different types of products: 1, 2, 3 and 4. Three resources are used in the production process: A , B and C . The amount of each resource used in the production of each product unit, the availability of the resources and the profit obtained from each product unit are given in the following table:

Resource	Products				Resource available
	1	2	3	4	
<i>A</i>	2	3	$\frac{3}{2}$	4	300
<i>B</i>	2	4	3	1	500
<i>C</i>	5	1	2	2	250
Profit	4	3	6	2	

Let x_j be the number of units of product j produced, $j = 1, 2, 3, 4$. We formulate the following linear model to represent the production problem described.

$$\begin{aligned} \max z &= 4x_1 + 3x_2 + 6x_3 + 2x_4 \\ \text{subject to} \\ 2x_1 + 3x_2 + \frac{3}{2}x_3 + 4x_4 &\leq 300 \\ 2x_1 + 4x_2 + 3x_3 + x_4 &\leq 500 \\ 5x_1 + x_2 + 2x_3 + 2x_4 &\leq 250 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Let us suppose now that another enterprise wants to purchase all of the firm's resources, $b_1 = 300$, $b_2 = 500$ and $b_3 = 250$. This enterprise has to determine the price to pay for a unit of each of them. Consequently, decision-variables y_1 , y_2 and y_3 are defined as: the price to pay for a unit of each of the resources *A*, *B* and *C*, respectively. The objective of the enterprise is to minimize the total cost of purchasing all of them.

$$\min G = 300y_1 + 500y_2 + 250y_3$$

Resource prices must be set high enough to encourage the firm to sell its resources to the other enterprise. Otherwise, the firm would have no reason to sell them, and would prefer to use them to produce the products and obtain the corresponding benefits. In other words, to produce a unit of product 1, the firm consumes 2 units of resource *A*, 2 units of resource *B*, and 5 units of resource *C*. If those units of resource are sold to the second enterprise, the firm will receive $2y_1 + 2y_2 + 5y_3$ monetary units; that is, the so called *economic cost* of production of product 1. Since the firm obtains a profit of 4 from each unit of product 1 produced, the enterprise will have to pay at least 4 units for the combination

of resources required to produce a unit of product 1. Therefore, the enterprise must choose the values of y_1 , y_2 and y_3 to satisfy the following constraint (the first constraint of the dual problem):

$$2y_1 + 2y_2 + 5y_3 \geq 4$$

Similar reasoning is followed for the other three products, and that is how we obtain the following dual model:

$$\begin{aligned} \min G &= 300y_1 + 500y_2 + 250y_3 \\ \text{subject to} \\ 2y_1 + 2y_2 + 5y_3 &\geq 4 \\ 3y_1 + 4y_2 + y_3 &\geq 3 \\ \frac{3}{2}y_1 + 3y_2 + 2y_3 &\geq 6 \\ 4y_1 + y_2 + 2y_3 &\geq 2 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

In summary, when the primal model is in maximization symmetric form, the variables of the dual are related to the value of the resources available. For this reason, the dual variables are often referred to as *shadow prices*.

It is also interesting to note that the previous discussion shows very clearly that the i th dual variable does indeed correspond to the i th primal constraint.

Let us now use the concept of the economic cost of the primal variables in order to give an economic interpretation of the simplex method. A nonbasic variable x_j of the primal model can be chosen to enter the basis in a subsequent iteration of the simplex algorithm, if its reduced cost coefficient $z_j - c_j$ is negative. The reason for this condition is that $z_j - c_j$ is precisely the j th dual constraint; when the reduced cost coefficient $z_j - c_j$ is negative, the economic cost of the production of j is lower than the profit associated with its production, and thus, the production of j becomes profitable. This reasoning can be verified in the following:

$$z_j - c_j = \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{a}_j - c_j = \mathbf{y}^T \mathbf{a}_j - c_j = \sum_{i=1}^m a_{ij} y_i - c_j.$$

Therefore, if $z_j - c_j < 0$ holds,

$$\sum_{i=1}^m a_{ij} y_i < c_j,$$

which means that, the economic cost of producing j is lower than the profit c_j associated with its production.

To conclude this section, we can say that the simplex method examines all the nonbasic variables in the current basic feasible solution to see which one can provide a more profitable use of the resources. If none of them can, the current solution is optimal.

3.6 The dual simplex method

In Chapter 2 we saw that the identity matrix is selected as the initial basis to compute a basic feasible solution to the primal problem using the simplex algorithm, and that if necessary, artificial variables will be introduced to the model. In subsequent iterations, the algorithm moves from a basic feasible solution to an improved basic feasible solution of the primal problem, until optimality is reached, that is, until $z_j - c_j \geq 0$ holds in the tableau, for all j . Such optimality condition is related to the feasibility of the dual problem. In fact, we can say that the simplex algorithm starts at a primal feasible solution, and stops when dual feasibility is reached. From this point onward, this method will be called the *simplex primal* method.

In this section we describe the *dual simplex* algorithm. This algorithm also starts selecting the identity matrix \mathbf{I} as the initial basis, which in this case is always formed by slack variables. The first step is to write the model in maximization symmetric form and to add a slack variable to each constraint. If the initial tableau is dual feasible, then iterations of the algorithm will be performed until the primal feasibility, keeping dual feasibility, is reached (if the problem is feasible). If the initial tableau constructed for the first canonical basis is not dual feasible, then an artificial constraint will be added to the primal problem to obtain the dual feasibility, as we will see in Section 3.7.

3.6.1 The dual simplex algorithm

The objective is to maximize. Find an initial basis $\mathbf{B} = \mathbf{I}$ of slack variables.

Step 1. Construct the initial tableau, where $z_j - c_j \geq 0$ for all j .

Step 2. With regard to primal feasibility, there are two cases to consider.

- If $x_{Bi} \geq 0$, $i = 1, \dots, m$, the current **solution is optimal**. Stop.

- If there exists $x_{Bi} < 0$, then the dual solution may be improved. Go to Step 3.

Step 3. Basis modification.

- Select a vector to leave the basis, according to the following criteria:

$$x_{Br} = \min_i \{ x_{Bi} / x_{Bi} < 0 \}$$

The r th row is the **pivot row**.

- Select a vector to enter the basis, according to the following criteria:

$$\frac{z_k - c_k}{y_{rk}} = \max_j \left\{ \frac{z_j - c_j}{y_{rj}} / y_{rj} < 0 \right\}$$

The k th column is the **pivot column**. y_{rk} is the **pivot element**.

If $y_{rj} \geq 0$ for all j , the **problem is infeasible**. Stop.

Step 4. Compute the new tableau by pivoting in the same way as stated in the simplex algorithm. Go to Step 2.

Example. Consider the following linear problem:

$$\min z = 3x_1 + 2x_2$$

subject to

$$x_1 + 2x_2 \geq 3$$

$$-2x_1 + x_2 \geq 2$$

$$x_1 + 4x_2 \geq 7$$

$$x_1, x_2 \geq 0$$

We put the problem in maximization symmetric form before applying the dual simplex algorithm.

$$\max (-z) = -3x_1 - 2x_2$$

subject to

$$-x_1 - 2x_2 \leq -3$$

$$2x_1 - x_2 \leq -2$$

$$-x_1 - 4x_2 \leq -7$$

$$x_1, x_2 \geq 0$$

We introduce three slack variables to put the three constraints in equation form. The three slack variables provide the initial canonical basis.

$$\begin{aligned} \max (-z) &= -3x_1 - 2x_2 + 0x_3 + 0x_4 + 0x_5 \\ \text{subject to} \\ -x_1 - 2x_2 + x_3 &= -3 \\ 2x_1 - x_2 + x_4 &= -2 \\ -x_1 - 4x_2 + x_5 &= -7 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

- **Step 1.** Construct the initial tableau. All $z_j - c_j$ are nonnegative; the tableau is dual feasible.

	x_1	x_2	x_3	x_4	x_5	
	3	2	0	0	0	$-\frac{1}{2}$
\mathbf{a}_3	-1	-2	1	0	0	$-\frac{1}{2}$
\mathbf{a}_4	2	-1	0	1	0	$\frac{1}{4}$
\mathbf{a}_5	-1	-4	0	0	1	-7

- **Step 2.** The tableau is not primal feasible. Then, the dual solution may be improved.

$$\mathbf{x}_B = \begin{pmatrix} -3 \\ -2 \\ -7 \end{pmatrix} \not\geq \mathbf{0}$$

- **Step 3.** Basis modification.

$$x_{B3} = \min\{-3, -2, -7\} = -7$$

Vector \mathbf{a}_5 leaves the basis. The 3rd row is the pivot-row.

$$\frac{z_k - c_k}{y_{ik}} = \max \left\{ \frac{3}{-1}, \frac{2}{-4} \right\} = -\frac{1}{2} = \frac{z_2 - c_2}{y_{22}}$$

Vector \mathbf{a}_2 enters the basis. The 2nd column is the pivot-column.

The pivot element is -4 .

- **Step 4.** Compute the new tableau. The first row multiplier is $\frac{1}{2}$, the second row multiplier is $\frac{1}{4}$ and the multiplier for row zero is $-\frac{1}{2}$.

After the pivoting operations are used, we get the tableau shown below:

		x_1	x_2	x_3	x_4	x_5		
		$\frac{5}{2}$	0	0	0	$\frac{1}{2}$	$-\frac{7}{2}$	-2
0	\mathbf{a}_3	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	2
0	\mathbf{a}_4	$\frac{9}{4}$	0	0	1	$-\frac{1}{4}$	$-\frac{1}{4}$	
-2	\mathbf{a}_2	$\frac{1}{4}$	1	0	0	$-\frac{1}{4}$	$\frac{7}{4}$	1

We go to Step 2 and see that the tableau is not primal feasible. Therefore, we need to proceed with another iteration of the dual simplex algorithm. Vector \mathbf{a}_4 leaves the basis and \mathbf{a}_5 enters it. We get this tableau:

		x_1	x_2	x_3	x_4	x_5	
		7	0	0	2	0	-4
0	\mathbf{a}_3	-4	0	1	-2	0	1
0	\mathbf{a}_5	-9	0	0	-4	1	1
-2	\mathbf{a}_2	-2	1	0	-1	0	2

Back at Step 2, we now verify that the tableau is primal feasible, and thus, it is optimal. The optimal solution to the problem is $x_1^* = 0$ and $x_2^* = 2$, and the optimal objective value is $z^* = 4$. \square

3.7 The artificial constraint technique

The dual simplex algorithm requires a dual feasible solution in the initial tableau. However, for some linear problems such initial tableau is not readily obtainable. The artificial constraint technique is used to extend the application of the dual simplex algorithm to solve all linear models.

When the initial tableau is not dual feasible, the technique consists in adding an artificial constraint to the model that is being solved. The introduction of such artificial constraint should not modify the feasible region of the problem. The

aim of introducing it is to obtain the dual feasibility in the tableau. Then, the dual simplex algorithm can be carried out on the modified problem. The artificial constraint added to the model is:

$$\sum_{j \in N} x_j \leq M.$$

where N is the set of variables such that the reduced cost coefficients $z_j - c_j$ are negative in the initial tableau. To guarantee that the original feasible region of the problem is not modified by the introduction of the artificial constraint to the model, M must be positive and sufficiently large.

Example. Consider the following linear model:

$$\begin{aligned} \max z &= x_1 + 6x_2 \\ \text{subject to} \\ x_1 + 2x_2 &\leq 20 \\ x_1 + \frac{1}{2}x_2 &\geq \frac{1}{2} \\ x_1, x_2 &\geq 0 \end{aligned}$$

We transform it into the maximization symmetric form, and add a slack variable to each of the constraints.

$$\begin{aligned} \max z &= x_1 + 6x_2 + 0x_3 + 0x_4 \\ \text{subject to} \\ x_1 + 2x_2 + x_3 &= 20 \\ -x_1 - \frac{1}{2}x_2 + x_4 &= -\frac{1}{2} \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

The initial canonical basis is $\mathbf{B} = \mathbf{I} = (\mathbf{a}_3, \mathbf{a}_4)$. We form the following initial tableau:

	x_1	x_2	x_3	x_4	
	-1	-6	0	0	0
\mathbf{a}_3	1	2	1	0	20
\mathbf{a}_4	-1	$-\frac{1}{2}$	0	1	$-\frac{1}{2}$

Since the initial tableau is not dual feasible, an artificial constraint must be introduced before applying the dual simplex algorithm. We add the following artificial constraint to the model:

$$x_1 + x_2 \leq M.$$

We add a slack variable to the artificial constraint, which leads to the following model:

$$\begin{aligned} \max z &= x_1 + 6x_2 + 0x_3 + 0x_4 + 0x_5 \\ \text{subject to} \\ x_1 + 2x_2 + x_3 &= 20 \\ -x_1 - \frac{1}{2}x_2 + x_4 &= -\frac{1}{2} \\ x_1 + x_2 + x_5 &= M \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

$\mathbf{B} = \mathbf{I} = (\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5)$ is now the initial canonical basis, and we form the initial tableau corresponding to the model with the artificial constraint.

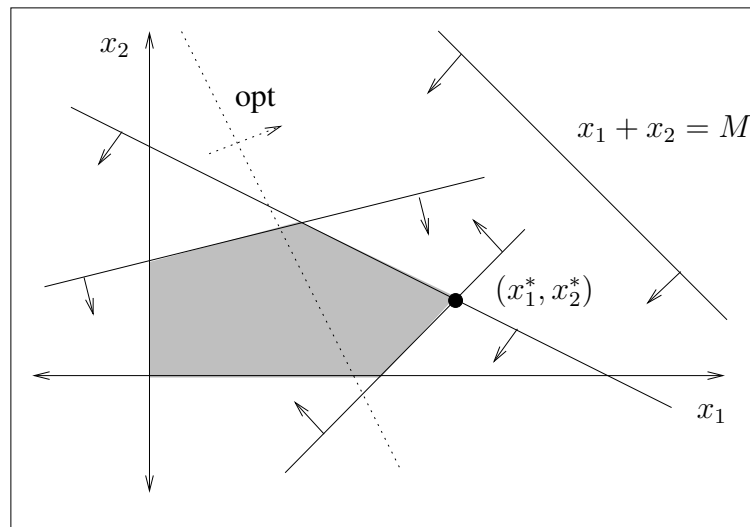
		x_1	x_2	x_3	x_4	x_5	
		-1	-6	0	0	0	-6
0	\mathbf{a}_3	1	2	1	0	0	20
0	\mathbf{a}_4	-1	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$
0	\mathbf{a}_5	1	1	0	0	1	M

Later on, we will see how to select the entering and the leaving vectors in such initial tableau to ensure dual feasibility in the subsequent tableau. \square

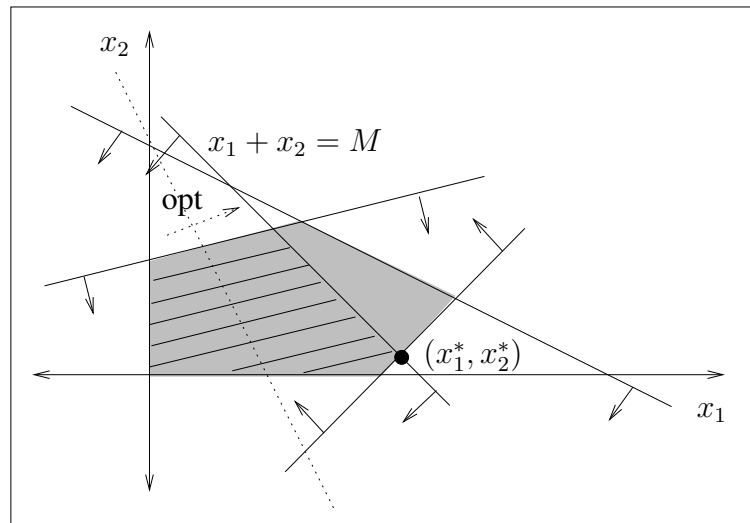
3.7.1 The effect of the artificial constraint

We can briefly say that, the artificial constraint technique extends the use of the dual simplex algorithm when necessary. Nevertheless, the introduction of the artificial constraint to the model should never modify the feasible region of the original model; to ensure this, M must be positive and as large as necessary.

The figure below illustrates the introduction of an artificial constraint with a large enough M , such that it causes no effect on the feasible region, that is to say, the shaded polygon in the figure.

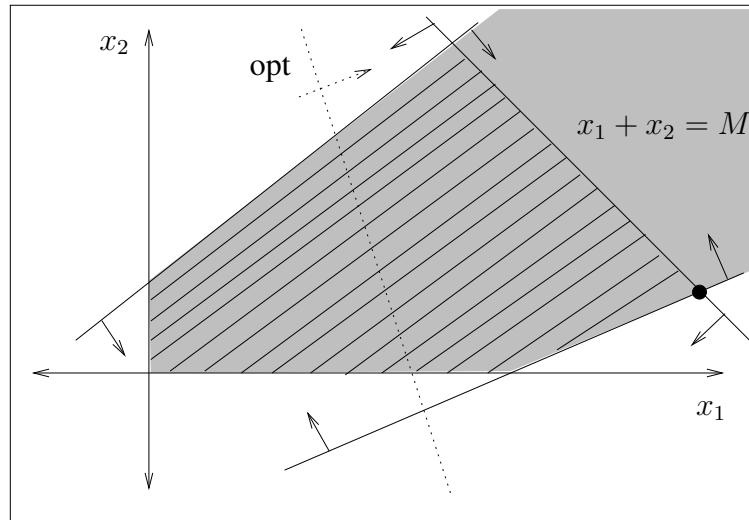


However, if M is not large enough, some points in the feasible region are discarded, because the artificial constraint eliminates them. Look at the illustration below to see that the feasible solutions in the shaded polygon not included in the area marked with parallel lines are not in the feasible region of the model with the artificial constraint $x_1 + x_2 \leq M$.



Note also that, when the feasible region of the problem is unbounded, the artificial constraint bounds it, even if M is large. In this last case, the artificial constraint limits the primal solution; the primal optimal solution verifies the equality

of the artificial constraint. An illustration of this case is shown below, where it can be seen that the primal optimal solution lies on the straight line that represents the artificial constraint equality.



As a conclusion to the two preceding cases, we can say that, when the slack variable of the artificial constraint is nonbasic in the optimal solution, we can conclude that either the value of M is not large enough, or the problem is unbounded.

3.7.2 The dual simplex algorithm with artificial constraint

The objective is to maximize. Find an initial basis $\mathbf{B} = \mathbf{I}$ of slack variables.

Step 1. Construct the initial tableau.

Step 2. With regard to the dual feasibility, there are two cases to consider.

- If $z_j - c_j \geq 0$ holds for all j , then go to Step 3.
- If there exists at least a column such that $z_j - c_j < 0$, then add an **artificial constraint** to the model and construct the initial tableau for the model with the additional constraint. Select a vector to enter the basis according to the following criteria:

$$z_k - c_k = \min_j \{ z_j - c_j / z_j - c_j < 0 \}.$$

Column k is the pivot column. Select the slack vector of the artificial constraint to leave the basis; the corresponding row is the pivot row. Compute the new tableau by pivoting in the same way as stated in the simplex algorithm. The new tableau is dual feasible. Go to Step 3.

Step 3. Primal feasibility.

- If the model has no additional artificial constraint,
 - If $x_{Bi} \geq 0$ for all i , then the solution is **optimal**. Stop.
 - If there exists at least a row i such that $x_{Bi} < 0$, then the solution may be improved. Go to Step 4.
- If the model has an additional artificial constraint,
 - If $x_{Bi} \geq 0$ for all i , and the slack variable of the artificial constraint is basic and positive, then the solution is **optimal**. Stop.
 - If $x_{Bi} \geq 0$ for all i , and the slack variable of the artificial constraint is nonbasic, or it is basic and has value zero, then the problem is **unbounded**. Stop.
 - If there exists at least a row i such that $x_{Bi} < 0$, then the solution may be improved. Go to Step 4.

Step 4. Basis modification.

- Select a vector to leave the basis, according to the following criteria:

$$x_{Br} = \min_i \{ x_{Bi} / x_{Bi} < 0 \}.$$

The r th row is the **pivot row**.

- Select a vector to enter the basis, according to the following criteria:

$$\frac{z_k - c_k}{y_{rk}} = \max_j \left\{ \frac{z_j - c_j}{y_{rj}} / y_{rj} < 0 \right\}.$$

The k th column is the **pivot column**. y_{rk} is the **pivot element**. Go to Step 5.

If $y_{rj} \geq 0$ for all j in the pivot row, the problem is **infeasible**. Stop.

Step 5. Compute the new tableau by pivoting in the same way as stated in the simplex algorithm. Go to Step 3.

3.8 Some illustrative examples

In this section, we solve three linear models which can only be solved by the dual simplex algorithm after the artificial constraint technique has been applied. We interpret the tableau for a feasible problem, an infeasible problem and an unbounded problem.

Example. (A feasible problem.) Let us consider the example of page 118. As we have previously seen, the initial tableau for the canonical basis constituted by slack variables is not dual feasible. Therefore, we need to add the artificial constraint $x_1 + x_2 \leq M$ to obtain the dual feasibility, before applying the dual simplex algorithm.

The initial tableau is displayed below (see page 118).

		x_1	x_2	x_3	x_4	x_5	
		-1	-6	0	0	0	0
0	\mathbf{a}_3	1	2	1	0	0	20
0	\mathbf{a}_4	-1	$-\frac{1}{2}$	0	1	0	$-\frac{1}{2}$
0	\mathbf{a}_5	1	1	0	0	1	M

We select vector \mathbf{a}_2 to enter the basis because $z_2 - c_2$ is the minimum reduced cost coefficient between the two negative $z_j - c_j$. We select the slack vector of the artificial constraint, \mathbf{a}_5 , to leave the basis. After pivoting operations are used, we get the following tableau:

		x_1	x_2	x_3	x_4	x_5	
		5	0	0	0	6	$6M$
0	\mathbf{a}_3	-1	0	1	0	-2	$-2M + 20$
0	\mathbf{a}_4	$-\frac{1}{2}$	0	0	1	$\frac{1}{2}$	$\frac{1}{2}M - \frac{1}{2}$
6	\mathbf{a}_2	1	1	0	0	1	M

It can be verified that the tableau is dual feasible, and as a consequence, the dual simplex algorithm can now be carried out normally. Vector \mathbf{a}_3 is selected to leave the basis and vector \mathbf{a}_5 to enter it. After pivoting, we get:

		x_1	x_2	x_3	x_4	x_5	
		2	0	3	0	0	60
0	\mathbf{a}_5	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	1	$M - 10$
0	\mathbf{a}_4	$-\frac{3}{4}$	0	$\frac{1}{4}$	1	0	$\frac{9}{2}$
6	\mathbf{a}_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	10

Since $\mathbf{x}_B \geq \mathbf{0}$ and the slack variable of the artificial constraint is basic and positive, $x_5^* = M - 10$, the solution is optimal for the original linear model (the model without the additional artificial constraint). The optimal solution is $x_1^* = 0$ and $x_2^* = 10$, and the optimal objective value is $z^* = 60$. \square

Example. (An unbounded problem.) Consider the following linear model:

$$\begin{aligned} \max z &= -4x_1 + 5x_2 \\ \text{subject to} \\ 2x_1 + 2x_2 &\geq 4 \\ x_1 - x_2 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

We transform it into the maximization symmetric form and add the needed slack variables and the artificial constraint $x_2 \leq M$, which leads to:

$$\begin{aligned} \max z &= -4x_1 + 5x_2 + 0x_3 + 0x_4 + 0x_5 \\ \text{subject to} \\ -2x_1 - 2x_2 + x_3 &= -4 \\ -x_1 + x_2 + x_4 &= -3 \\ x_2 + x_5 &= M \\ x_1, x_2, x_3, x_4, x_5 &\geq 0 \end{aligned}$$

Applying the dual simplex algorithm, we obtain the following series of tableaux:

		x_1	x_2	x_3	x_4	x_5		
		4	-5	0	0	0	0	
0	\mathbf{a}_3	-2	-2	1	0	0	-4	-2
0	\mathbf{a}_4	-1	1	0	1	0	-3	1
0	\mathbf{a}_5	0	1	0	0	1	M	
		4	0	0	0	5	$5M$	
0	\mathbf{a}_3	-2	0	1	0	2	$2M - 4$	2
0	\mathbf{a}_4	-1	0	0	1	-1	$-M - 3$	
5	\mathbf{a}_2	0	1	0	0	1	M	0
		0	0	0	4	1	$M - 12$	
0	\mathbf{a}_3	0	0	1	-2	4	$4M + 2$	
-4	\mathbf{a}_1	1	0	0	-1	1	$M + 3$	
5	\mathbf{a}_2	0	1	0	0	1	M	

The last tableau is optimal, because it is primal feasible, $\mathbf{x}_B \geq \mathbf{0}$, and dual feasible, $z_j - c_j \geq 0$ for all j . However, the slack vector of the artificial constraint, \mathbf{a}_5 , is not basic. This leads to conclude that the problem is unbounded. \square

Example. (An infeasible problem.) Consider the following linear model:

$$\begin{aligned} \max z &= 2x_1 + x_2 \\ \text{subject to} \\ x_1 + x_2 &\leq 2 \\ -3x_1 + x_2 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

We transform it into the maximization symmetric form and add the needed slack variables and the artificial constraint $x_1 + x_2 \leq M$, which leads to:

$$\max z = 2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5$$

subject to

$$x_1 + x_2 + x_3 = 2$$

$$3x_1 - x_2 + x_4 = -3$$

$$x_1 + x_2 + x_5 = M$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Below are displayed the tableaux obtained after several iterations of the dual simplex algorithm.

		x_1	x_2	x_3	x_4	x_5		
		-2	-1	0	0	0	0	
0	\mathbf{a}_3	1	1	1	0	0	2	1
0	\mathbf{a}_4	3	-1	0	1	0	-3	3
0	\mathbf{a}_5	1	1	0	0	1	M	
		0	1	0	0	2	$2M$	
0	\mathbf{a}_3	0	0	1	0	-1	$-M + 2$	0
0	\mathbf{a}_4	0	-4	0	1	-3	$-3M - 3$	
2	\mathbf{a}_1	1	1	0	0	1	M	$-\frac{1}{4}$
		0	0	0	$\frac{1}{4}$	$\frac{5}{4}$	$\frac{5}{4}M - \frac{3}{4}$	
0	\mathbf{a}_3	0	0	1	0	-1	$-M + 2$	
1	\mathbf{a}_2	0	1	0	$-\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}M + \frac{3}{4}$	$-\frac{3}{4}$
2	\mathbf{a}_1	1	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}M - \frac{3}{4}$	$-\frac{1}{4}$

		x_1	x_2	x_3	x_4	x_5	
		0	0	$\frac{5}{4}$	$\frac{1}{4}$	0	$\frac{7}{4}$
0	\mathbf{a}_5	0	0	-1	0	1	$M - 2$
1	\mathbf{a}_2	0	1	$\frac{3}{4}$	$-\frac{1}{4}$	0	$\frac{9}{4}$
2	\mathbf{a}_1	1	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{4}$

Since $x_{B3} < 0$, the third row is the pivot row, and vector \mathbf{a}_1 is selected to leave the tableau. Nevertheless, $y_{3j} \geq 0$ for all columns in the pivot row, which means that, according to the dual simplex algorithm, there is no pivot element. This leads to the conclusion that the problem is infeasible because it has not been possible to find a feasible solution $\mathbf{x}_B \geq \mathbf{0}$ in the tableau. \square

