Chapter 2

The simplex method

The graphical solution can be used to solve linear models defined by using only two or three variables. In Chapter 1 the graphical solution of two variable linear models has been analyzed. It is not possible to solve linear models of more than three variables by using the graphical solution, and therefore, it is necessary to use an algebraic procedure. In 1949, George B. Dantzig developed the simplex method for solving linear programming problems.

The simplex method is designed to be applied only after the linear model is expressed in the following form:

Standard form. A linear model is said to be in standard form if all constraints are equalities, and each one of the values in vector **b** and all variables in the model are nonnegative.

$$\max(\min)z = \mathbf{c}^T\mathbf{x}$$
$$\text{subject to}$$
$$\mathbf{Ax} = \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

If the objective is to maximize, then it is said that the model is in maximization standard form. Otherwise, the model is said to be in minimization standard form.

2.1 Model manipulation

Linear models need to be written in standard form to be solved by using the simplex method. By simple manipulations, any linear model can be transformed into

an equivalent one written in standard form. The objective function, the constraints and the variables can be transformed as needed.

1. **The objective function.**

   Computing the minimum value of function z is equivalent to computing the maximum value of function $-z$,

   $$\min \ z = \sum_{j=1}^{n} c_j x_j \quad \Longleftrightarrow \quad \max \ (-z) = \sum_{j=1}^{n} -c_j x_j$$

   For instance, $\min \ z = 3x_1 - 5x_2$ and $\max \ (-z) = -3x_1 + 5x_2$ are equivalent; values of variables x_1 and x_2 that make the value of z minimum and $-z$ maximum are the same. It holds $\min \ z = -\max \ (-z)$.

2. **Constraints.**

   (a) The ith constraint of a linear model can be multiplied by -1, like this:

   $$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad \Longleftrightarrow \quad \sum_{j=1}^{n} -a_{ij} x_j \leq -b_i$$

   For instance, if we multiply the inequality constraint $2x_1 + 3x_2 \leq -2$ by -1, we obtain the equivalent $-2x_1 - 3x_2 \geq 2$ inequality constraint.

   (b) Inequality constraints can be converted to equality constraints.

   $$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \Longleftrightarrow \quad \sum_{j=1}^{n} a_{ij} x_j + y = b_i$$

   $$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad \Longleftrightarrow \quad \sum_{j=1}^{n} a_{ij} x_j - y = b_i$$

   Variable y used in the previous transformations is called *slack variable*; in both cases, it is assumed that $y \geq 0$.

   For example, constraints $x_1 - 4x_2 \leq 4$ and $x_1 - 4x_2 + y = 4$ are equivalent, if $y \geq 0$ holds.

(c) An equality constraint can be transformed into two inequality constraints, by proceeding like this:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \iff \sum_{j=1}^{n} a_{ij}x_j \le b_i \quad \text{and} \quad \sum_{j=1}^{n} a_{ij}x_j \ge b_i$$

For example, the equality constraint $-2x_1 + x_2 = 2$ is equivalent to having both the following two inequality constraints: $-2x_1 + x_2 \le 2$ and $-2x_1 + x_2 \ge 2$.

3. **Variables.**

Variables need to be nonnegative. These are the transformations that can be used in case they are not.

- If a variable x_j is defined to be negative, $x_j \le 0$, then the substitution $x_j = -x'_j$ produces a nonnegative variable x'_j.
- If a variable x_j is unrestricted in sign, then it can be replaced by $x_j = x'_j - x''_j$, where $x'_j \ge 0$ and $x''_j \ge 0$.
  - If $x'_j > x''_j$, then $x_j > 0$ holds.
  - If $x'_j < x''_j$, then $x_j < 0$ holds.
  - If $x'_j = x''_j$, then $x_j = 0$ holds.

Example. Taking into account the equivalences presented previously, let us transform the following linear model into the maximization standard form:

$$\begin{aligned}
\min \quad z \ &= x_1 + 2x_2 + x_3 \\
\text{subject to} \quad & \\
x_1 + x_2 - x_3 &\ge 2 \\
x_1 - 2x_2 + 5x_3 &\le -1 \\
x_1 + x_2 + x_3 &= 4 \\
x_1 \ge 0 \ , \ x_2 \le 0, \ x_3 &: \text{unrestricted}
\end{aligned}$$

- Objective function.

  We transform the objective function into a maximization function:

$$\max (-z) \ = -x_1 - 2x_2 - x_3$$

- Constraints.

  The first inequality constraint can be written as an equality constraint by subtracting the nonnegative slack variable x_4,

  $$x_1 + x_2 - x_3 - x_4 = 2.$$

  We add the slack variable x_5 to the second constraint,

  $$x_1 - 2x_2 + 5x_3 + x_5 = -1.$$

  We multiply by -1 the constraint, so that the right-hand-side value becomes positive,

  $$-x_1 + 2x_2 - 5x_3 - x_5 = 1.$$

  The third constraint is appropriately written:

  $$x_1 + x_2 + x_3 = 4.$$

- Variables.

  Variable x_1 is defined nonnegative. Variable x_2 is not defined nonnegative, $x_2 \leq 0$, thus we define variable x_2' as $x_2' = -x_2$, and change the sign of coefficients of variable x_2 in the model when we replace it by x_2'. Variable x_3 is unrestricted in sign. Therefore, we define $x_3 = x_3' - x_3''$ and replace it in the model.

The resulting linear model is in maximization standard form and may be written as:

$$\max\,(-z)\ = -x_1 + 2x_2' - x_3' + x_3'' + 0x_4 + 0x_5$$
$$\text{subject to}$$
$$x_1 - x_2' - x_3' + x_3'' - x_4 = 2$$
$$-x_1 - 2x_2' - 5x_3' + 5x_3'' - x_5 = 1$$
$$x_1 - x_2' + x_3' - x_3'' = 4$$
$$x_1, x_2', x_3', x_3'', x_4, x_5 \geq 0$$

$\square$

The addition or subtraction of slack variables should not change the objective function. Thus, their coefficient in the objective function is zero.

2.2 Solving linear models

As we have seen in the previous section, linear models can always be written in standard form. The simplex algorithm that we study in this chapter is design to maximize; therefore, the maximization standard form is used onwards. Remember that in the standard form, vector $\mathbf{b}$ is defined as nonnegative.

Consider the following linear model in standard form:

$$\max \quad z = \mathbf{c}^T \mathbf{x}$$
$$\text{subject to}$$
$$\mathbf{Ax} = \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

where $\mathbf{x}$ and $\mathbf{c}$ are $n \times 1$ vectors, $\mathbf{b}$ is an $m \times 1$ vector and $\mathbf{A}$ is an $m \times n$ matrix.

Suppose that $m < n$, and that the rank of matrix $\mathbf{A}$ is m. Systems of this kind have an infinite number of solutions. The problem consists in computing the solution that optimizes the objective function.

Next, we give some definitions needed to develop the simplex method.

1. **Definition**. If vector $\mathbf{x}$ satisfies a system of linear equations $\mathbf{Ax} = \mathbf{b}$, then it is said to be a *solution* to the system.

2. **Definition**. A solution vector $\mathbf{x}$, is said to be *feasible* if $\mathbf{x} \geq 0$.

3. **Definition**. Let $\mathbf{B}$ be a basic matrix constituted by extracting m columns out of matrix $\mathbf{A}$. $\mathbf{x}_B$ is called a *basic solution* if $\mathbf{Bx}_B = \mathbf{b}$ is hold. The components of $\mathbf{x}_B$ are called *basic variables*. In a basic solution, all nonbasic variables are zero, $\mathbf{x}_N = \mathbf{0}$ (see Appendix A). Thus, a basic solution has a maximum of m components different from zero.

   Moreover, if all components of vector $\mathbf{x}_B$ are nonnegative, $\mathbf{x}_B \geq \mathbf{0}$,then it is called a *basic feasible solution*.

4. **Definition**. A basic feasible solution is said to be *degenerate* if at least one component of $\mathbf{x}_B$ is zero. If $\mathbf{x}_B > \mathbf{0}$, then it is called *nondegenerate*.

5. **Definition**. The set of all feasible solutions of a linear problem is called the *feasible region* or the *convex set of feasible solutions*, and it is noted by F.

6. **Definition**. Notation $\mathbf{x}^*$ is used to denote the *optimal solution* of a linear model, and $z^* = \mathbf{c}^T\mathbf{x}^*$ to denote the *optimal objective value*.

7. **Definition**. A linear model is said to be *unbounded* if there exists no finite optimal value for the objective function, that is, if $z^* \to +\infty$ or $z^* \to -\infty$.

8. **Definition**. If a linear model has more than one optimal solution then it is said that the model has *multiple optimal solutions*.

From now on, each time we mention the basis, we will be referring to the basic matrix.

Example. Consider the following linear model in standard form.

$$\max \ z \ = 3x_1 + 6x_2 + 5x_3 + 4x_4 + x_5$$
$$\text{subject to}$$
$$2x_1 + 8x_2 + 3x_3 + x_4 + x_5 = 6$$
$$x_1 + \ x_2 \ + 2x_3 + x_4 \qquad = 4$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

where

$$\mathbf{c}^T = (3, \ 6, \ 5, \ 4, \ 1), \quad \mathbf{x}^T = (x_1, \ x_2, \ x_3, \ x_4, \ x_5)$$

$$\mathbf{A} = \begin{pmatrix} 2 & 8 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

The rank of matrix $\mathbf{A}$ is 2, lower than the number of variables in the system of equations. Therefore, the system of equations has an infinite number of solutions. However, the number of basic solutions is finite. Remember that to compute all the basic solutions, we have to choose all the possible bases $\mathbf{B}$ among the columns of matrix $\mathbf{A}$ (see Appendix A), and solve the following system:

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b},$$

where $\mathbf{x}_B$ is the vector of basic variables. Next, we compute three basic solutions.

1. Matrix $\mathbf{B}$, constituted by the first and the fourth columns of matrix $\mathbf{A}$, is a basis.

$$\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Therefore,

$$\mathbf{x}_B = \begin{pmatrix} x_1 \\ x_4 \end{pmatrix}, \quad \mathbf{N} = \begin{pmatrix} 8 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_N = \begin{pmatrix} x_2 \\ x_3 \\ x_5 \end{pmatrix}$$

Hence,

$$\mathbf{x}_B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 8 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_5 \end{pmatrix} \quad (2.1)$$

By assigning different values to nonbasic variables x_2, x_3 and x_5, the infinite solutions of the system may be computed. Next, we compute two of them.

- If we assign $x_2 = 0$, $x_3 = 0$ and $x_5 = 0$ in equation (2.1), operations yield the following basic solution:

$$\mathbf{x}_B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

  The basic solution obtained ($x_1 = 2$, $x_4 = 2$) is feasible, because there is no negative component.

- If we assign $x_2 = 0$, $x_3 = 1$ and $x_5 = 0$ in equation (2.1), operations yield the following solution:

$$\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 7 & 1 & 1 \\ -6 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Vector $\mathbf{x}^T = (1,\ 0,\ 1,\ 1,\ 0)$ is a solution to the system of equations. It is feasible, because all components are nonnegative. However, it is not a basic solution because the nonbasic variable x_3 has a value different from zero.

2. If we choose matrix $\mathbf{B}$, constituted by the first two columns of matrix $\mathbf{A}$,

$$\mathbf{B} = \begin{pmatrix} 2 & 8 \\ 1 & 1 \end{pmatrix}$$

which is a basis, and if we set all nonbasic variables to zero, then we get the following basic solution:

$$\mathbf{x}_B = \begin{pmatrix} 2 & 8 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} & \frac{4}{3} \\ \frac{1}{6} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{13}{3} \\ -\frac{1}{3} \end{pmatrix}$$

The basic solution computed is not feasible, because variable $x_2 = -\frac{1}{3}$ is negative.

3. If we choose the matrix constituted by the third and the fifth columns of $\mathbf{A}$,

$$\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$$

which is a basis, and if we set all nonbasic variables to zero, then we get the following basic solution:

$$\mathbf{x}_B = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

The basic solution computed is feasible. In addition, it is degenerate because there exists a basic variable that has a value of zero in the solution: $x_5 = 0$.

$$\square$$

2.3 Extreme points and basic feasible solutions

As we have seen in the graphical solution of linear models, the optimal solution can be found in an extreme point of the convex set of feasible solutions. If the

problem has multiple optimal solutions, at least one of them is an extreme point of the set. It is not possible to solve linear models with more than three variables by using the graphical solution, and therefore, an algebraic method is needed. In the following two theorems, we demonstrate some results that will allow us to go from the geometric solution method to the algebraic method. In the proves, the following results are used:

1. The set F of feasible solutions of a linear model written in standard form is a *closed convex set*.

2. A point $\mathbf{x}$ in the convex set F is an *extreme point* if there do not exist two points $\mathbf{x}_1$ and $\mathbf{x}_2$ in F, $\mathbf{x}_1 \neq \mathbf{x}_2$, such that $\mathbf{x} = \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$, for some $0 < \lambda < 1$.

3. Every point $\mathbf{x}$ in a closed and bounded convex set can be expressed by a convex linear combination of the extreme points, as follows:

$$\mathbf{x} = \sum_{i=1}^{q} \lambda_i \mathbf{x}_i \quad , \quad 0 \leq \lambda_i \leq 1 \quad , \quad \sum_{i=1}^{q} \lambda_i = 1$$

Every extreme point corresponds to a basic feasible solution, and conversely, every basic feasible solution corresponds to an extreme point, as it is proved in the following theorem:

Theorem 2.3.1 *Consider the following linear model in standard form:*

$$\begin{aligned} \max \ z = \ & \mathbf{c}^T \mathbf{x} \\ subject \ to \ & \\ & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

$\mathbf{x}$ *is a basic feasible solution if and only if* $\mathbf{x}$ *is an extreme point of* F.

Proof.
$\boxed{\Rightarrow}$ If $\mathbf{x}$ is a *basic feasible solution*, then let us prove that it is an extreme point.

If $\mathbf{x}$ is a basic feasible solution, then it has at most m components greater than zero. In order to simplify the notation, we assume that they are the first m components. Then,

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{0} \end{pmatrix}$$

Le us assume that there are two points $\mathbf{x}_1, \mathbf{x}_2$ in F, $\mathbf{x}_1 \neq \mathbf{x}_2$, such that,

$$\mathbf{x} = \lambda\mathbf{x}_1 + (1 - \lambda)\mathbf{x}_2 \ , \ \ 0 < \lambda < 1.$$

Consider

$$\mathbf{x}_1 = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}'_1 \end{pmatrix} \ \ \text{and} \ \ \mathbf{x}_2 = \begin{pmatrix} \mathbf{y}_2 \\ \mathbf{y}'_2 \end{pmatrix}$$

where vectors $\mathbf{y}_i$, ($i = 1, 2$) are $m \times 1$, and vectors $\mathbf{y}'_i$, ($i = 1, 2$) are $(n - m) \times 1$. Then, from the equation:

$$\begin{pmatrix} \mathbf{x}_B \\ \mathbf{0} \end{pmatrix} = \lambda \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}'_1 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} \mathbf{y}_2 \\ \mathbf{y}'_2 \end{pmatrix}$$

we can conclude:

$$\mathbf{0} = \lambda\mathbf{y}'_1 + (1 - \lambda)\mathbf{y}'_2.$$

$\mathbf{y}'_1, \mathbf{y}'_2 \geq \mathbf{0}$, $\lambda > 0$ and $1 - \lambda > 0$. Therefore, $\mathbf{y}'_1 = \mathbf{y}'_2 = \mathbf{0}$ holds. Thus, solutions $\mathbf{x}$, $\mathbf{x}_1$ and $\mathbf{x}_2$ are basic solutions, and, as they correspond to the same basis, $\mathbf{x}_B = \mathbf{x}_1 = \mathbf{x}_2$ is satisfied.

As a result, the assumption made was wrong; there do not exist such points $\mathbf{x}_1$ and $\mathbf{x}_2$ in F, and consequently, $\mathbf{x}$ is an extreme point.

$\boxed{\Leftarrow}$ If $\mathbf{x}$ is an *extreme point*, then it is a basic feasible solution.

Assume that vector $\mathbf{x}$ has k components greater than zero, and the rest are zero. Then, the system of constraints can be written as follows:

$$\sum_{i=1}^{k} x_i\mathbf{a}_i = \mathbf{b}.$$

To show that $\mathbf{x}$ is a basic feasible solution, it must be shown that vectors $\mathbf{a}_i$, $i = 1, \ldots, k$, are linearly independent. We do this by contradiction. Suppose $\mathbf{a}_i$,

$i = 1, \ldots, k$, are linearly dependent. Then, there is a nontrivial linear combination that is zero:

$$\sum_{i=1}^{k} \alpha_i \mathbf{a}_i = \mathbf{0}.$$

We define:

$$\alpha^T = (\alpha_1, \ldots, \alpha_k, 0, \ldots, 0).$$

It is possible to select ϵ such that $\mathbf{x}_1 = \mathbf{x} + \epsilon\alpha \geq \mathbf{0}$ and $\mathbf{x}_2 = \mathbf{x} - \epsilon\alpha \geq \mathbf{0}$. Moreover, $\mathbf{x}_1 \neq \mathbf{x}_2$ holds. We then have:

$$\mathbf{x} = \frac{1}{2}\mathbf{x}_1 + \frac{1}{2}\mathbf{x}_2.$$

This cannot occur, since $\mathbf{x}$ is an extreme point of F. Thus, $\mathbf{a}_i$, $i = 1, \ldots, k$, are linearly independent and $\mathbf{x}$ is a basic feasible solution. $\qquad\square$

An optimal solution to a linear model is an extreme point of the feasible region, as it is proved in the following theorem.

Theorem 2.3.2 *Consider the following linear model in standard form:*

$$\begin{aligned} \max \quad z \quad &= \mathbf{c}^T\mathbf{x} \\ \textit{subject to} \quad & \\ \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

The optimal value of the objective function is obtained at an extreme point of the feasible region F.

Proof. Suppose we have an optimal solution $\mathbf{x}^*$ which is not an extreme point, and that $z^* = \mathbf{c}^T\mathbf{x}^*$ is the optimal objective value of the model. Consequently, for every point $\mathbf{x}$ in the feasible region F, the following holds:

$$z = \mathbf{c}^T\mathbf{x} \leq \mathbf{c}^T\mathbf{x}^* = z^*$$

Let us consider the set of all the extreme points of F, $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$. Every point in the feasible region F, and therefore $\mathbf{x}^*$ as well, can be expressed by a convex linear combination of the extreme points as follows:

$$\mathbf{x}^* = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i \ , \quad \lambda_i \geq 0, \quad \sum_{i=1}^{k} \lambda_i = 1$$

Thus,

$$z^* = \mathbf{c}^T \mathbf{x}^* = \mathbf{c}^T \sum_{i=1}^{k} \lambda_i \mathbf{x}_i = \sum_{i=1}^{k} \lambda_i \mathbf{c}^T \mathbf{x}_i$$

Since $\quad \max_i (\mathbf{c}^T \mathbf{x}_i) \geq \mathbf{c}^T \mathbf{x}_i, \ i = 1, \ldots, k,$

$$z^* = \sum_{i=1}^{k} \lambda_i \mathbf{c}^T \mathbf{x}_i \leq \sum_{i=1}^{k} \lambda_i \max_i (\mathbf{c}^T \mathbf{x}_i) = \max_i (\mathbf{c}^T \mathbf{x}_i) \sum_{i=1}^{k} \lambda_i = \max_i (\mathbf{c}^T \mathbf{x}_i) \leq z^*.$$

As a result, $z^* = \max_i (\mathbf{c}^T \mathbf{x}_i)$, and it can be concluded that the optimal objective value is obtained in an extreme point. $\qquad \square$

To illustrate the results proved in Theorem 2.3.1 and Theorem 2.3.2, we consider an example. As linear models possess a finite number of basic solutions at most, it is possible to develop an iterative procedure that, starting at a basic feasible solution moves towards a better basic feasible solution, until an optimal solution is reached.

Example. Consider the following linear model:

$$\begin{aligned} \max \quad z \ &= \ x_1 + 2x_2 \\ \text{subject to} \quad & \\ -x_1 + 4x_2 \ &\leq \ 4 \\ x_1 - x_2 \ &\leq \ 3 \\ x_1, x_2 \ &\geq \ 0 \end{aligned}$$

We can represent graphically the feasible region and compute the extreme points. There are four extreme points in F. By moving the objective function line over the feasible region in the optimization direction, we reach the extreme point B.

$$O = (0, \ 0), \quad A = (0, \ 1), \quad B = \left(\frac{16}{3}, \ \frac{7}{3} \right), \quad C = (3, \ 0)$$
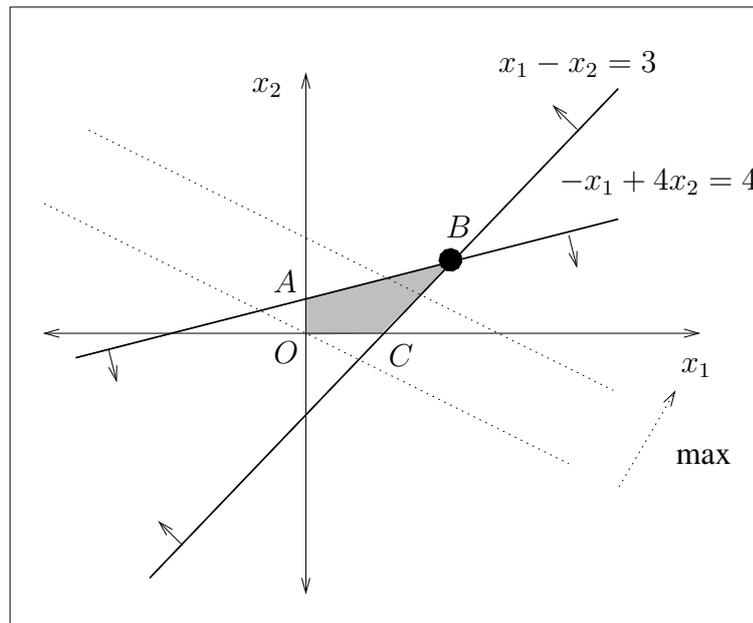
To compute algebraically the basic solutions, and to see the correspondence between the extreme points of the feasible region and the basic feasible solutions, we must introduce the slack variables x_3 and x_4. This yields the following equivalent linear model in standard form:

$$\max \quad z \; = x_1 + 2x_2 + 0x_3 + 0x_4$$

subject to

$$-x_1 + 4x_2 \;\; +x_3 \qquad\quad = 4$$
$$x_1 - x_2 \qquad\quad +x_4 \; = 3$$
$$x_1, \; x_2, \; x_3, \; x_4 \geq 0$$

Matrix $\mathbf{A}$ in the system of linear equations has four columns. Therefore, we can extract at most 6 bases $\mathbf{B}$ out of matrix $\mathbf{A}$; combining the four columns and grouping two by two the linearly independent, we have at most 6 bases. Let us examine the 6 options mentioned.

1. Basis $\mathbf{B} = (\mathbf{a}_1 \; \mathbf{a}_2)$.

$$\mathbf{x}_B = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{4}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{16}{3} \\ \frac{7}{3} \end{pmatrix}$$

This basic solution is feasible, and it corresponds to the extreme point B shown in the graphical representation.

2. Basis $\mathbf{B} = (\mathbf{a}_1 \ \mathbf{a}_3)$.

$$\mathbf{x}_B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

This basic solution is feasible, and it corresponds to the extreme point C shown in the graphical representation.

3. Basis $\mathbf{B} = (\mathbf{a}_1 \ \mathbf{a}_4)$.

$$\mathbf{x}_B = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}$$

This basic solution is not feasible, because there are negative components. Thus, this basic solution does not correspond to any extreme point in the graphical representation.

4. Basis $\mathbf{B} = (\mathbf{a}_2 \ \mathbf{a}_3)$.

$$\mathbf{x}_B = \begin{pmatrix} 4 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 16 \end{pmatrix}$$

This basic solution is not feasible, because there are negative components. Thus, this basic solution does not correspond to any extreme point in the graphical representation.

5. Basis $\mathbf{B} = (\mathbf{a}_2 \ \mathbf{a}_4)$.

$$\mathbf{x}_B = \begin{pmatrix} 4 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ \frac{1}{4} & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

This basic solution is feasible, and it corresponds to the extreme point A shown in the graphical representation.

6. Basis $\mathbf{B} = (\mathbf{a}_3 \; \mathbf{a}_4)$.

$$\mathbf{x}_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

This basic solution is feasible, and it corresponds to the extreme point O shown in the graphical representation.

$\square$

All the sets of two vectors extracted out of $\mathbf{A}$ in the previous example are bases, and we computed six basic solutions. If the vectors chosen happen to be linearly dependent, then they do not constitute a basis, and therefore, it is not possible to compute a basic solution from them. The fourth and the fifth basic solutions computed do not correspond to any extreme point of the feasible region, because they are not feasible.

2.4 The simplex method

As we have seen in the graphical solution of linear models, there are different types of optimal solutions: unique optimal solution, multiple optimal solutions, unbounded solution. In this section, we determine the conditions that must hold to identify each one of them, and develop an iterative procedure to solve linear models: the simplex algorithm.

2.4.1 Definitions and notation

To begin with, we will first see some basic definitions and establish the notation used in the development of the linear programming theory. Consider a linear model in standard form.

$$\begin{aligned} \max \; z \; &= \; \mathbf{c}^T \mathbf{x} \\ \text{subject to} \\ \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

Suppose that there are m linearly independent rows and n columns in matrix $\mathbf{A}$, $n > m$. Let $\mathbf{B}$ be an $m \times m$ basis formed with m linearly independent columns

of $\mathbf{A}$, and let $\mathbf{N}$ be the remaining columns of $\mathbf{A}$. To simplify the notation, we assume that the columns chosen to extract basis $\mathbf{B}$ are the first ones of matrix $\mathbf{A}$. The basic components of vectors $\mathbf{c}$ and $\mathbf{x}$ are denoted by $\mathbf{c}_B$ and $\mathbf{x}_B$, respectively; nonbasic components are denoted by $\mathbf{c}_N$ and $\mathbf{x}_N$. The linear model can be written as:

$$\max\ z = (\mathbf{c}_B^T \mid \mathbf{c}_N^T) \begin{pmatrix} \mathbf{x}_B \\ - \\ \mathbf{x}_N \end{pmatrix}$$

subject to

$$(\mathbf{B} \mid \mathbf{N}) \begin{pmatrix} \mathbf{x}_B \\ - \\ \mathbf{x}_N \end{pmatrix} = \mathbf{b}$$

$$\mathbf{x}_B, \mathbf{x}_N \geq \mathbf{0}$$

And we get:

$$\max\ z = \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N$$

subject to

$$\mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}$$

$$\mathbf{x}_B, \mathbf{x}_N \geq \mathbf{0}$$

- *Basic solution*. By setting $\mathbf{x}_N = \mathbf{0}$, the system of linear equations is $\mathbf{B}\mathbf{x}_B = \mathbf{b}$, and the basic solution is computed,

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b},$$

where

$$\mathbf{x}_B = \begin{pmatrix} x_{B1} \\ x_{B2} \\ \vdots \\ x_{Bm} \end{pmatrix}$$

- *Objective value.* Let $\mathbf{c}_B^T = (c_{B1}, c_{B2}, \ldots, c_{Bm})$,

$$z = \mathbf{c}_B^T \mathbf{x}_B = (c_{B1}, c_{B2}, \ldots, c_{Bm}) \begin{pmatrix} x_{B1} \\ x_{B2} \\ \vdots \\ x_{Bm} \end{pmatrix} = \sum_{i=1}^{m} c_{Bi} x_{Bi}.$$

- *Coordinate vector.* Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be the column vectors of matrix $\mathbf{A}$. Each vector can be written as a linear combination of vectors in the basis $\mathbf{B}$. The notation used is the following:

$$\mathbf{a}_j = y_{1j}\mathbf{a}_1 + y_{2j}\mathbf{a}_2 + \cdots + y_{mj}\mathbf{a}_m = \sum_{i=1}^{m} y_{ij}\mathbf{a}_i.$$

The coordinate vector of $\mathbf{a}_j$ is:

$$\mathbf{y}_j = \begin{pmatrix} y_{1j} \\ y_{2j} \\ \vdots \\ y_{mj} \end{pmatrix}$$

The coordinate vector is computed by solving the system $\mathbf{a}_j = \mathbf{B}\mathbf{y}_j$, that is,

$$\mathbf{y}_j = \mathbf{B}^{-1}\mathbf{a}_j.$$

- *Reduced cost coefficients* $z_j - c_j$. The scalar value z_j for each vector $\mathbf{a}_j$ is computed as follows:

$$z_j = \mathbf{c}_B^T \mathbf{y}_j = c_{B1}y_{1j} + c_{B2}y_{2j} + \cdots + c_{Bm}y_{mj} = \sum_{i=1}^{m} c_{Bi}y_{ij}.$$

Example. Consider the following linear model in standard form:

$$\begin{aligned} \max \quad z \ &= \ 3x_1 + 4x_2 + 5x_3 + 6x_4 \\ \text{subject to} \quad & \\ 2x_1 + x_2 + x_3 + 8x_4 &= 6 \\ x_1 + x_2 + 2x_3 + x_4 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

Consider the basis constituted by the first two columns of matrix $\mathbf{A}$. The linear model can be written in matrix notation like this:

$$\max \ z \ = \ (3, 4 \mid 5, 6) \begin{pmatrix} x_1 \\ x_2 \\ - \\ x_3 \\ x_4 \end{pmatrix}$$

subject to

$$\begin{pmatrix} 2 & 1 & 1 & 8 \\ 1 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ - \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

• Basic solution.

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

• Objective value. Since $\mathbf{c}_B^T = (3, 4)$,

$$z = \mathbf{c}_B^T \mathbf{x}_B = (3, \ 4) \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 14$$

• The coordinate vector of a nonbasic vector. Let us compute it for vector $\mathbf{a}_4$.

$$\begin{pmatrix} 8 \\ 1 \end{pmatrix} = y_{14} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + y_{24} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_{14} \\ y_{24} \end{pmatrix}$$

By solving the system, we get:

$$\mathbf{y}_4 = \begin{pmatrix} y_{14} \\ y_{24} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 8 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ -6 \end{pmatrix}$$

- Reduced cost coefficient $z_4 - c_4$:

$$z_4 - c_4 = \mathbf{c}_B^T \mathbf{y}_4 - c_4 = (3, \ 4) \begin{pmatrix} 7 \\ -6 \end{pmatrix} - 6 = -3 - 6 = -9.$$

<div align="right">□</div>

2.4.2   Improvement of a basic feasible solution

When seeking an optimal solution for a linear model, it is only necessary to consider the basic feasible solutions defined by the constraints, because the optimal objective value is always achieved at a basic feasible solution. The following theorem states the conditions for improvement when computing a basic feasible solution from another basic feasible solution. Starting at a basic feasible solution, the simplex method will make use of these conditions to move from a basic feasible solution to a better one, until the optimality condition holds.

Theorem 2.4.1 (Improvement of a basic feasible solution) *Consider a linear model in standard form:*

$$\begin{aligned} \max \ z \ &= \mathbf{c}^T \mathbf{x} \\ \textit{subject to} \ & \\ \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

Let $\mathbf{B}$ *be a basis formed by columns of* $\mathbf{A}$, *and let* $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ *be the basic feasible solution relative to* $\mathbf{B}$, *and* $z = \mathbf{c}_B^T \mathbf{x}_B$ *the corresponding objective value.*

If there exists a nonbasic vector $\mathbf{a}_j$ *in matrix* $\mathbf{A}$ *such that* $z_j - c_j < 0$ *and for such vector* $\mathbf{a}_j$ *there exists a positive coordinate* y_{ij}, $i = 1, \ldots, m$, *then there exists another basic feasible solution* $\overset{\wedge}{\mathbf{x}}_B$, *where*

$$\hat{z} = \overset{\wedge}{\mathbf{c}}_B^T \overset{\wedge}{\mathbf{x}}_B \geq \ z \ = \ \mathbf{c}_B^T \mathbf{x}_B.$$

Proof. To simplify notation, we assume that basis $\mathbf{B}$ is formed by extracting the first m columns out of matrix $\mathbf{A}$, that is, $\mathbf{B} = (\mathbf{a}_1 \ \ldots \ \mathbf{a}_r \ \ldots \ \mathbf{a}_m)$. Since $\mathbf{x}_B$ is a basic feasible solution, it satisfies:

$$x_{B1}\mathbf{a}_1 + x_{B2}\mathbf{a}_2 + \cdots + x_{Bm}\mathbf{a}_m = \mathbf{b} = \sum_{i=1}^{m} x_{Bi}\mathbf{a}_i. \tag{2.2}$$

Addends with nonbasic variables have not been included in the equation.

Vectors $\mathbf{a}_{m+1}, \mathbf{a}_{m+2}, \ldots, \mathbf{a}_n$ are not basic. Any of them can be expressed as a linear combination of the basic vectors.

$$\mathbf{a}_j = \sum_{i=1}^{m} y_{ij}\mathbf{a}_i \ , \ j = m+1, \ldots, n \tag{2.3}$$

Since $\mathbf{a}_j \neq \mathbf{0}$, $j = m+1, \ldots, n$, at least one coordinate y_{ij} is nonzero. Let us assume it is the $y_{rj} \neq 0$. If we write separately the rth addend in equation (2.3), we get:

$$\mathbf{a}_j = \sum_{\substack{i=1 \\ i \neq r}}^{m} y_{ij}\mathbf{a}_i + y_{rj}\mathbf{a}_r \tag{2.4}$$

We can express vector $\mathbf{a}_r$ as a linear combination of vectors $\mathbf{a}_1, \ldots, \mathbf{a}_j, \ldots, \mathbf{a}_m$. The new set of vectors obtained by replacing vector $\mathbf{a}_r$ by $\mathbf{a}_j$ constitutes a new basis (see Theorem A.3.2 on page 248).

$$\overset{\wedge}{\mathbf{B}} = (\mathbf{a}_1 \ \ldots \ \mathbf{a}_j \ \ldots \ \mathbf{a}_m).$$

Let us now see how we can compute the new basic solution $\overset{\wedge}{\mathbf{x}}_B$. If we solve vector $\mathbf{a}_r$ in equation (2.4), we obtain:

$$\mathbf{a}_r = \frac{1}{y_{rj}}\mathbf{a}_j - \sum_{\substack{i=1 \\ i \neq r}}^{m} \frac{y_{ij}}{y_{rj}}\mathbf{a}_i.$$

Writing separately the rth addend in equation (2.2), we get:

$$\sum_{\substack{i=1 \\ i \neq r}}^{m} x_{Bi}\mathbf{a}_i + x_{Br}\mathbf{a}_r = \mathbf{b}.$$

Substituting vector $\mathbf{a}_r$, we have:

$$\sum_{\substack{i=1 \\ i \neq r}}^{m} x_{Bi}\mathbf{a}_i + x_{Br}\left[\frac{1}{y_{rj}}\mathbf{a}_j - \sum_{\substack{i=1 \\ i \neq r}}^{m} \frac{y_{ij}}{y_{rj}}\mathbf{a}_i\right] = \mathbf{b}.$$

After rearranging the addends,

$$\sum_{\substack{i=1 \\ i \neq r}}^{m}\left(x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}}\right)\mathbf{a}_i + \frac{x_{Br}}{y_{rj}}\mathbf{a}_j = \mathbf{b},$$

we obtain a new set of m values which satisfies the set of equations, and consequently it constitutes the new basic solution.

$$\overset{\wedge}{\mathbf{x}}_B = \begin{cases} x_{Bi} - x_{Br}\dfrac{y_{ij}}{y_{rj}} & i \neq r \\ \dfrac{x_{Br}}{y_{rj}} & i = r \end{cases} \tag{2.5}$$

The question remaining is to find out under which conditions the new basic solution $\overset{\wedge}{\mathbf{x}}_B$ will be feasible. Every component in $\overset{\wedge}{\mathbf{x}}_B$ must be greater than or equal to zero for it to be feasible, that is,

- For the rth component, $\hat{x}_{Br} = \dfrac{x_{Br}}{y_{rj}} \geq 0$.

  As $y_{rj} \neq 0$ and $x_{Br} \geq 0$, then the condition $y_{rj} > 0$ must be satisfied to maintain feasibility, that is, $\dfrac{x_{Br}}{y_{rj}} \geq 0$.

- For the remaining, $\hat{x}_{Bi} = x_{Bi} - x_{Br}\dfrac{y_{ij}}{y_{rj}} \geq 0, \; i = 1, \ldots, m, \; i \neq r$

  As $x_{Bi} \geq 0$, $x_{Br} \geq 0$ and $y_{rj} > 0$, there are three cases to consider:

  - If $y_{ij} < 0$, then $\hat{x}_{Bi} = x_{Bi} - x_{Br}\dfrac{y_{ij}}{y_{rj}} \geq 0$ holds.

  - If $y_{ij} = 0$, then $\hat{x}_{Bi} = x_{Bi} - x_{Br}\dfrac{y_{ij}}{y_{rj}} = x_{Bi} \geq 0$ holds.

  - If $y_{ij} > 0$, then to maintain feasibility, that is, $\hat{x}_{Bi} \geq 0$, the following condition must be satisfied:
  $$\hat{x}_{Bi} = x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}} \geq 0 \iff \frac{x_{Bi}}{y_{ij}} - \frac{x_{Br}}{y_{rj}} \geq 0 \iff \frac{x_{Bi}}{y_{ij}} \geq \frac{x_{Br}}{y_{rj}}.$$

Taking into account all the foregoing conditions, vector $\mathbf{a}_r$ which is replaced by another in the basis, is determined by the following criteria:

$$\frac{x_{Br}}{y_{rj}} = \min_{i}\{\frac{x_{Bi}}{y_{ij}} \; / \; y_{ij} > 0\} \tag{2.6}$$

Up to now, it has been proved that vector $\mathbf{a}_r$ which satisfies criteria (2.6) can be replaced in the basis to compute a new basic feasible solution. Let us now determine under which conditions the new basic feasible solution $\overset{\wedge}{\mathbf{x}}_B$ is better than the old one, $\mathbf{x}_B$, that is, under which conditions the following holds:

$$\hat{z}=\overset{\wedge}{\mathbf{c}}_B^T\overset{\wedge}{\mathbf{x}}_B\geq \mathbf{c}_B^T\mathbf{x}_B = z.$$

We compute $\hat{z}$, and write separately the rth addend,

$$\hat{z}= \sum_{i=1}^{m} \hat{c}_{Bi}\hat{x}_{Bi}= \sum_{\substack{i=1 \\ i\neq r}}^{m} \hat{c}_{Bi}\hat{x}_{Bi} + \hat{c}_{Br}\hat{x}_{Br} \; .$$

By substituting $\hat{c}_{Bi}$, $\hat{c}_{Br}$, $\hat{x}_{Bi}$ and $\hat{x}_{Br}$, we get:

$$\hat{z}= \sum_{\substack{i=1 \\ i\neq r}}^{m} c_{Bi}(x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}}) + c_j\frac{x_{Br}}{y_{rj}} = (*)$$

When $i = r$, the following holds:

$$c_{Br}(x_{Br} - x_{Br}\frac{y_{rj}}{y_{rj}}) = 0.$$

Therefore, we can ignore the condition $i \neq r$ in the summation, which leads to the following expression:

$$(*) = \sum_{i=1}^{m} c_{Bi}(x_{Bi} - x_{Br}\frac{y_{ij}}{y_{rj}}) + c_j\frac{x_{Br}}{y_{rj}}.$$

By applying elementary straightforward operations,

$$\hat{z}= \sum_{i=1}^{m} c_{Bi}x_{Bi} - \frac{x_{Br}}{y_{rj}} \sum_{i=1}^{m} c_{Bi}y_{ij} + c_j\frac{x_{Br}}{y_{rj}} = z - \frac{x_{Br}}{y_{rj}}(z_j - c_j)$$

we obtain:

$$\hat{z} = z - \frac{x_{Br}}{y_{rj}}(z_j - c_j). \tag{2.7}$$

Hence,

$$\hat{z} \geq z \quad \Longleftrightarrow \quad -\frac{x_{Br}}{y_{rj}}(z_j - c_j) \geq 0 \quad \Longleftrightarrow \quad \frac{x_{Br}}{y_{rj}}(z_j - c_j) \leq 0.$$

As $x_{Br} \geq 0$ and $y_{rj} > 0$, to satisfy the relation $\hat{z} \geq z$, it is clear that $z_j - c_j < 0$ must necessarily hold.

We conclude that for vector $\mathbf{a}_j$, the one that will substitute vector $\mathbf{a}_r$ in the original basis, $z_j - c_j < 0$ holds. Usually, vector $\mathbf{a}_k$ whose reduced cost coefficient is the minimum among all the negative ones is chosen to enter the basis. Therefore, this is the rule followed to choose the entering vector $\mathbf{a}_k$:

$$z_k - c_k = \min_j \{z_j - c_j \ / \ z_j - c_j < 0\} \tag{2.8}$$

$\square$

2.4.3 Selection of entering and leaving vectors

According to the above theorem, we need to choose a basis and compute its corresponding basic feasible solution and objective value; afterwards, the replacement of a vector in the original basis by a nonbasic one results in a new basis. The corresponding basic solution is feasible and better than the previous one if the entering vector and the leaving one are selected using the criteria (2.6) and (2.8), which can be summarized as follows:

- *Entering vector rule*. Vector $\mathbf{a}_k$ is selected to enter the basis if the following rule holds:

$$\boxed{z_k - c_k = \min_j \{z_j - c_j \ / \ z_j - c_j < 0\}}$$

  The rule guarantees that the solution computed for the new basis will be better than the previous one, that is, $\hat{z} \geq z$.

- *Leaving vector rule*. Vector $\mathbf{a}_k$ enters the basis and will replace vector $\mathbf{a}_r$. The leaving vector $\mathbf{a}_r$ is selected according to the following rule:

$$\boxed{\frac{x_{Br}}{y_{rk}} = \min_i \left\{ \frac{x_{Bi}}{y_{ik}} \ / \ y_{ik} > 0 \right\}}$$

The rule guarantees that the solution computed for the new basis will be feasible, that is, $\hat{\mathbf{x}}_B \geq \mathbf{0}$.

2.4.4 Rules to compute $\hat{\mathbf{x}}_B$ and $\hat{z}$

From the proof of Theorem 2.4.1 and the rules for selecting the entering and leaving vectors, the calculation of the new basic feasible solution $\hat{\mathbf{x}}_B$ shown in (2.5), and the new objective value $\hat{z}$ shown in (2.7) can be written as follows:

- Calculation of the new basic feasible solution.

$$
\hat{\mathbf{x}}_B =
\begin{cases}
x_{Bi} - x_{Br}\dfrac{y_{ik}}{y_{rk}} & i \neq r \\[2mm]
\dfrac{x_{Br}}{y_{rk}} & i = r
\end{cases}
\tag{2.9}
$$

- Calculation of the new objective value.

$$
\hat{z} = z - \frac{x_{Br}}{y_{rk}}(z_k - c_k)
\tag{2.10}
$$

Example. Let us consider the following linear problem:

$$
\begin{aligned}
\max \quad z \;=\;& 4x_1 + 5x_2 + x_3 \\
\text{subject to} \quad & \\
& x_1 + x_2 + x_3 \leq 8 \\
-& x_1 - 2x_2 + x_3 \leq -2 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

First, we need to transform the model into the maximization standard form. Therefore, we introduce the slack variables x_4 and x_5 to transform inequality constraints into equalities. Moreover, we multiply the second constraint by -1 so that $\mathbf{b} \geq \mathbf{0}$ holds. This leads to the following model:

$$\max \; z = 4x_1 + 5x_2 + x_3 + 0x_4 + 0x_5$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 \; + x_4 \qquad\quad &= 8 \\ x_1 + 2x_2 - x_3 \qquad\quad -x_5 \;\; &= 2 \\ x_1, \; x_2, \; x_3, \; x_4, \; x_5 \geq 0 \end{aligned}$$

Next, we need to choose a basis; the one formed by extracting the first and the fourth columns out of matrix $\mathbf{A}$, for instance. The corresponding basic solution is computed like this:

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 8 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

The basic solution is feasible, and thus it lies in the feasible region. The corresponding objective value is computed as follows:

$$z = \mathbf{c}_B^T\mathbf{x}_B = (4, 0) \begin{pmatrix} 2 \\ 6 \end{pmatrix} = 8.$$

Let us analyze whether the basic feasible solution that has thus been computed can be improved.

Selection of the vector entering the basis. In order to select the entering vector, we need to compute the reduced cost coefficients $z_2 - c_2$, $z_3 - c_3$ and $z_5 - c_5$ for the nonbasic vectors $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_5$.

- Calculation of the reduced cost coefficient $z_2 - c_2$.

$$\mathbf{a}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} ; \; \mathbf{y}_2 = \mathbf{B}^{-1}\mathbf{a}_2 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$z_2 - c_2 = \mathbf{c}_B^T\mathbf{y}_2 - c_2 = (4, \, 0) \begin{pmatrix} 2 \\ -1 \end{pmatrix} - 5 = 3 \; > 0.$$

- Calculation of the reduced cost coefficient $z_3 - c_3$.

$$\mathbf{a}_3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \ ; \ \mathbf{y}_3 = \mathbf{B}^{-1}\mathbf{a}_3 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$z_3 - c_3 = (4, \, 0) \begin{pmatrix} -1 \\ 2 \end{pmatrix} - 1 = -5 \ < \ 0 \rightarrow \text{the solution can be improved.}$$

- Calculation of the reduced cost coefficient $z_5 - c_5$.

$$\mathbf{a}_5 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \ ; \ \mathbf{y}_5 = \mathbf{B}^{-1}\mathbf{a}_5 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$z_5 - c_5 = (4, 0) \begin{pmatrix} -1 \\ 1 \end{pmatrix} - 0 = -4 \ < \ 0 \rightarrow \text{the solution can be improved.}$$

There are two options to improve the solution: to introduce vector $\mathbf{a}_3$ in the basis or to introduce $\mathbf{a}_5$. We use the entering vector rule so that we can make a decision.

$$z_k - c_k = \min_{j}\{z_j - c_j \ / \ z_j - c_j < 0\} =$$

$$= \min\{z_3 - c_3 = -5 \ , \ z_5 - c_5 = -4\} = -5.$$

Consequently, vector $\mathbf{a}_3$ is selected to enter the basis.

Selection of the vector leaving the basis. In order to select the leaving vector, we need to consider vector $\mathbf{y}_3$ and the basic solution $\mathbf{x}_B$.

$$\mathbf{y}_3 = \begin{pmatrix} y_{13} \\ y_{23} \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \qquad \mathbf{x}_B = \begin{pmatrix} x_{B1} \\ x_{B2} \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

Vector $\mathbf{a}_r$, which satisfies the leaving rule, will be selected to leave the basis:

$$\frac{x_{Br}}{y_{r3}} = \min_{i}\left\{ \frac{x_{Bi}}{y_{i3}} \ / \ y_{i3} > 0 \right\} = \min\left\{ \frac{x_{B2}}{y_{23}} = \frac{6}{2} \right\} = 3.$$

The vector placed in the second position in the basis is selected to leave it, that is $\mathbf{a}_4$. This yields a new basis, a new basic feasible solution and a new objective value.

- By using rule (2.9), we compute the new solution.

$$\hat{x}_{B1} = x_{B1} - x_{B2}\frac{y_{13}}{y_{23}} = 2 - 6\left(\frac{-1}{2}\right) = 5.$$

$$\hat{x}_{B2} = \frac{x_{B2}}{y_{23}} = \frac{6}{2} = 3.$$

- By using rule (2.10), we compute the new objective value.

$$\hat{z} = z - \frac{x_{B2}}{y_{23}}(z_3 - c_3) = 8 - \frac{6}{2}(-5) = 23.$$

We can see that the new basic solution $\overset{\wedge}{\mathbf{x}}_B$ is feasible, and that it has improved the objective value. The same procedure may be applied repeatedly, until all the reduced cost coefficients become nonnegative. Once this condition holds, the solution will be optimal; the condition mentioned is established in the following theorem. $\qquad\square$

Theorem 2.4.2 (Optimality condition). *Consider a linear model in standard form:*

$$\begin{aligned} \max \ z \ &= \ \mathbf{c}^T\mathbf{x} \\ subject\ to \\ \mathbf{A}\mathbf{x} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

Let $\mathbf{B}$ be a basis formed by columns of $\mathbf{A}$, and let $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ be its corresponding basic feasible solution, and $z = \mathbf{c}_B^T\mathbf{x}_B$ its objective value. If $z_j - c_j$ is greater than or equal to zero for every vector $\mathbf{a}_j$ of matrix $\mathbf{A}$, then $\mathbf{x}_B$ is an optimal basic feasible solution for the model.

2.4.5 Unbounded solution

As we saw in the graphical solution to linear models, in some cases the optimal objective value may improve indefinitely. The following theorem states the necessary conditions for the solution to be unbounded.

Theorem 2.4.3 *Consider a linear model in standard form:*

$$\max \ z \ = \ \mathbf{c}^T \mathbf{x}$$

$$\textit{subject to}$$

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{x} \geq \mathbf{0}$$

Let $\mathbf{B}$ be a basis formed by columns of $\mathbf{A}$, and let $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ be its corresponding basic feasible solution, and $z = \mathbf{c}_B^T \mathbf{x}_B$ its objective value.

If there exists a nonbasic vector $\mathbf{a}_k$ in matrix $\mathbf{A}$ such that $z_k - c_k < 0$, and for such vector $\mathbf{a}_k$ all coordinates y_{ik} are less than or equal to zero, $i = 1, \ldots, m$, then the solution to the model is unbounded.

Proof. Let $\mathbf{x}_B$ be a basic feasible solution. As it is a solution, it satisfies the constraints:

$$x_{B1}\mathbf{a}_1 + x_{B2}\mathbf{a}_2 + \cdots + x_{Bm}\mathbf{a}_m = \mathbf{b}.$$

If there exists a nonbasic vector $\mathbf{a}_k$ in matrix $\mathbf{A}$ such that $z_k - c_k < 0$, then the objective value may be improved. But since $y_{ik} \leq 0$ holds, $i = 1, \ldots, m$, none of the basic vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$ can leave the basis so that a better basic feasible solution can be calculated. That is, none of them can be substituted by the entering vector $\mathbf{a}_k$. However, a new solution (although not a basic one) may be computed, which will make the objective value z take on arbitrarily large values.

Let us add and subtract $\theta \mathbf{a}_k$ on the left hand side of the previous equation, being θ any positive real value. This leads to:

$$x_{B1}\mathbf{a}_1 + x_{B2}\mathbf{a}_2 + \cdots + x_{Bm}\mathbf{a}_m - \theta\mathbf{a}_k + \theta\mathbf{a}_k = \mathbf{b}.$$

$$\sum_{i=1}^{m} x_{Bi}\mathbf{a}_i - \theta\mathbf{a}_k + \theta\mathbf{a}_k = \mathbf{b} \tag{2.11}$$

Vector $\mathbf{a}_k$ is nonbasic, and it can be written as a linear combination of the basic vectors:

$$\mathbf{a}_k = \sum_{i=1}^{m} y_{ik}\mathbf{a}_i.$$

In Equation (2.11), if we substitute vector $\mathbf{a}_k$ by its expression as a linear combination of the basic vectors, we get:

$$\sum_{i=1}^{m} x_{Bi}\mathbf{a}_i - \theta \sum_{i=1}^{m} y_{ik}\mathbf{a}_i + \theta\mathbf{a}_k = \mathbf{b},$$

which can also be written as follows:

$$\sum_{i=1}^{m}(x_{Bi} - \theta y_{ik})\mathbf{a}_i + \theta\mathbf{a}_k = \mathbf{b}.$$

Solutions calculated like this can have more than m components greater than zero, and thus, they are not basic solutions.

$$\hat{x} = \begin{pmatrix} x_{B1} - \theta y_{1k} \\ x_{B2} - \theta y_{2k} \\ \vdots \\ x_{Bm} - \theta y_{mk} \\ 0 \\ \vdots \\ \theta \\ \vdots \\ 0 \end{pmatrix} \tag{2.12}$$

It can be verified that it is a feasible solution. Since $\theta > 0$ holds, and as $x_{Bi} \geq 0$, $y_{ik} \leq 0$, $i = 1, \ldots, m$, then $x_{Bi} - \theta y_{ik} \geq 0$, $i = 1, \ldots, m$.

The objective value is calculated as follows:

$$\hat{z} = \sum_{i=1}^{m} c_{Bi}(x_{Bi} - \theta y_{ik}) + c_k\theta = \sum_{i=1}^{m} c_{Bi}x_{Bi} - \theta\sum_{i=1}^{m} c_{Bi}y_{ik} + c_k\theta =$$

$$= z - \theta z_k + \theta c_k = z - \theta(z_k - c_k)$$

Since $z_k - c_k < 0$ holds, the objective value $\hat{z}$ will increase according to the value of θ, and the solution to the model will be unbounded. □

Example. Let us compute the optimal solution to the following linear problem:

$$\max \ z \ = \ -3x_1 + 2x_2$$
$$\text{subject to}$$
$$x_1 - x_2 \le 5$$
$$2x_1 - 3x_2 \le 10$$
$$x_1, x_2 \ge 0$$

Writing the problem in standard form we have:

$$\max \ z = -3x_1 + 2x_2 + 0x_3 + 0x_4$$

$$\text{subject to}$$

$$x_1 \quad -x_2 \quad +x_3 \qquad \qquad = \ 5$$
$$2x_1 \quad -3x_2 \qquad \qquad +x_4 \ = \ 10$$
$$x_1, \ x_2, \ x_3, \ x_4 \ge 0$$

We select basis $\mathbf{B} = (\mathbf{a}_3 \ \mathbf{a}_4)$ and compute the corresponding basic solution.

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 5 \\ 10 \end{pmatrix}$$

The solution is feasible, and the objective value associated with it is the following:

$$z = \mathbf{c}_B^T \mathbf{x}_B = (0, \ 0) \begin{pmatrix} 5 \\ 10 \end{pmatrix} = 0.$$

Let us apply Theorem 2.4.1 to see if there is a better solution to the problem. In order to find out, we need to compute the reduced cost coefficients $z_1 - c_1$ and $z_2 - c_2$.

- The reduced cost coefficient $z_1 - c_1$.

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \ \mathbf{y}_1 = \mathbf{B}^{-1}\mathbf{a}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$z_1 - c_1 = \mathbf{c}_B^T \mathbf{y}_1 - c_1 = (0,\ 0) \begin{pmatrix} 1 \\ 2 \end{pmatrix} - (-3) = 3.$$

- The reduced cost coefficient $z_2 - c_2$.

$$\mathbf{a}_2 = \begin{pmatrix} -1 \\ -3 \end{pmatrix};\ \mathbf{y}_2 = \mathbf{B}^{-1}\mathbf{a}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

$$z_2 - c_2 = \mathbf{c}_B^T \mathbf{y}_2 - c_2 = (0,\ 0) \begin{pmatrix} -1 \\ -3 \end{pmatrix} - 2 = -2.$$

According to the entering vector rule, vector $\mathbf{a}_2$ is chosen to enter the basis.

$$\min_j \{ z_j - c_j\ /\ z_j - c_j < 0 \} = \min \{ z_2 - c_2 = -2 \} = -2$$

According to the leaving vector rule, we check vector $\mathbf{y}_2$ and find that all the components in the vector are negative, and thus, none of the basic vectors satisfy the leaving vector rule. Conditions established in Theorem 2.4.3 hold; consequently, the solution is unbounded.

As the original problem has only two decision variables, it can also be solved graphically. This allows the graphical representation of the unbounded solution.

$\square$
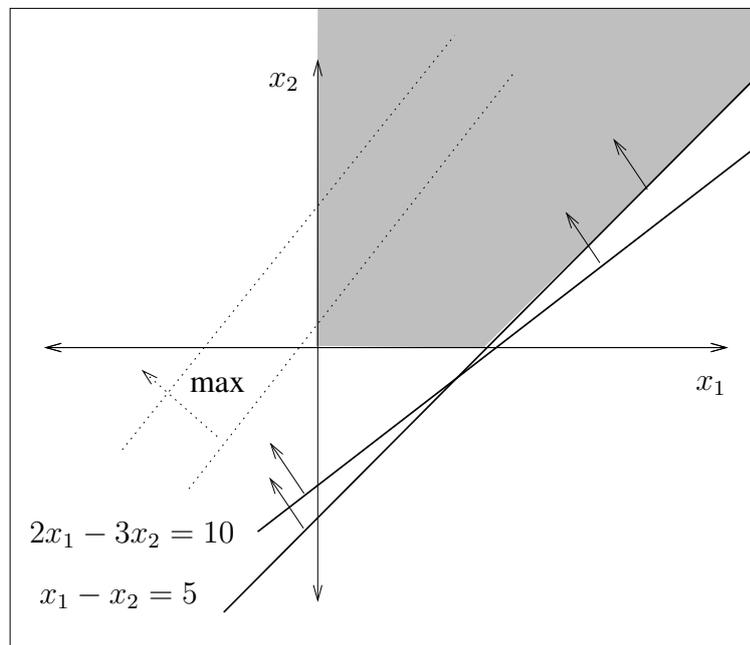
2.4.6   Multiple optimal solutions

When analyzing the graphical solution of linear models, we have already seen that some problems have more than one optimal solution. In such cases, it is said that the problem has *multiple optimal solutions*. This kind of solution may arise both with bounded or unbounded variables.

   The following theorem establishes the conditions under which a linear problem has multiple optimal solutions.

Theorem 2.4.4 *Consider a linear model in standard form:*

$$
\begin{aligned}
\max \quad z \quad &= \mathbf{c}^T \mathbf{x} \\
\textit{subject to} & \\
\mathbf{A}\mathbf{x} &= \mathbf{b} \\
\mathbf{x} &\geq \mathbf{0}
\end{aligned}
$$

   Let $\mathbf{B}$ be a basis formed by columns of $\mathbf{A}$, and let $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ be its corresponding basic feasible solution, and $z = \mathbf{c}_B^T \mathbf{x}_B$ its objective value.

   If $z_j - c_j \geq 0$ holds for every vector $\mathbf{a}_j$ of matrix $\mathbf{A}$, then the solution $\mathbf{x}_B$ is optimal. Moreover, if there exists a nonbasic vector $\mathbf{a}_k$ such that $z_k - c_k = 0$, and at least one coordinate $y_{ik} > 0$, $i = 1, \ldots, m$, then there exist multiple optimal solutions.

   Proof. Let $\mathbf{x}_B$ be a basic feasible solution. Since $z_j - c_j \geq 0$ holds for every vector $\mathbf{a}_j$ of matrix $\mathbf{A}$, from Theorem 2.4.2 we can say that $\mathbf{x}_B$ is optimal.

   If there exists a nonbasic vector $\mathbf{a}_k$ such that $z_k - c_k = 0$, and at least one coordinate $y_{ik} > 0$, $i = 1, \ldots, m$, then vector $\mathbf{a}_k$ can be selected to enter the basis to substitute a basic vector $\mathbf{a}_r$ that satisfies the leaving vector rule:

$$
\frac{x_{Br}}{y_{rk}} = \min_i \left\{ \frac{x_{Bi}}{y_{ik}} / y_{ik} > 0 \right\}
$$

   We then have a new basis $\overset{\wedge}{\mathbf{B}}$ and a new basic feasible solution $\overset{\wedge}{\mathbf{x}}_B$. The new objective value is computed as follows:

$$
\hat{z} = z - \frac{x_{Br}}{y_{rk}}(z_k - c_k) = z - 0 = z.
$$

As a result, $\overset{\wedge}{\mathbf{x}}_B$ is also optimal, because the objective value obtained for it is identical to the one obtained for $\mathbf{x}_B$, which has been proven to be optimal. $\quad\square$

Theorem 2.4.5 *Consider a linear model in standard form:*

$$\max \ z \ = \mathbf{c}^T \mathbf{x}$$
$$subject\ to$$
$$\mathbf{A}\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

Let $\mathbf{x}_1, \ldots, \mathbf{x}_p$ be optimal basic feasible solutions of the model. Then, every generalized convex linear combination of them is an optimal feasible solution to the model.

Proof. Let $\mathbf{x}$ be a vector obtained by any convex linear combination of the basic feasible solutions $\mathbf{x}_1, \ldots, \mathbf{x}_p$.

$$\mathbf{x} = \sum_{i=1}^{p} \mu_i \mathbf{x}_i\,, \quad \mu_i \geq 0, \quad i = 1, \ldots, p\,, \quad \sum_{i=1}^{p} \mu_i = 1$$

Let us prove that $\mathbf{x}$ is a solution which is also feasible and optimal.

1. $\mathbf{x}$ is a solution.

   Since each $\mathbf{x}_i$ is a solution, $i = 1, \ldots, p$, they verify $\mathbf{A}\mathbf{x}_i = \mathbf{b}$. It follows:

   $$\mathbf{A}\mathbf{x} = \mathbf{A}(\sum_{i=1}^{p} \mu_i \mathbf{x}_i) = \sum_{i=1}^{p} \mu_i \mathbf{A}\mathbf{x}_i = \mathbf{b} \sum_{i=1}^{p} \mu_i = \mathbf{b}.$$

   Therefore, $\mathbf{x}$ is a solution.

2. $\mathbf{x}$ is feasible.

   Since $\mathbf{x}_i \geq \mathbf{0}$ and $\mu_i \geq 0$, $i = 1, \ldots, p$, it follows:

   $$\mathbf{x} = \sum_{i=1}^{p} \mu_i \mathbf{x}_i \geq \mathbf{0}.$$

   Therefore, $\mathbf{x}$ is feasible.

3. $\mathbf{x}$ is optimal.

Each $\mathbf{x}_i$ is optimal, $i = 1, \ldots, p$, that is, $z^* = \mathbf{c}^T \mathbf{x}_i$. It follows:

$$\mathbf{c}^T \mathbf{x} = \mathbf{c}^T \sum_{i=1}^{p} \mu_i \mathbf{x}_i = \sum_{i=1}^{p} \mu_i \mathbf{c}^T \mathbf{x}_i = z^* \sum_{i=1}^{p} \mu_i = z^*.$$

Therefore, $\mathbf{x}$ is optimal.

□

Theorem 2.4.4 and Theorem 2.4.5 establish the conditions under which multiple optimal solutions for bounded variables are obtained. However, in some problems there are multiple optimal solutions for unbounded variables, such that the objective value has a bounded value. The following theorem establishes the conditions under which multiple optimal solutions are found for unbounded variables.

Theorem 2.4.6 *Let us consider a linear model in standard form:*

$$\max \quad z \;\; = \mathbf{c}^T \mathbf{x}$$
$$subject\ to$$
$$\mathbf{A}\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

Let $\mathbf{B}$ *be a basis formed by columns of* $\mathbf{A}$*, and let* $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ *be its corresponding basic feasible solution, and* $z = \mathbf{c}_B^T \mathbf{x}_B$ *its objective value. If* $z_j - c_j \geq 0$ *holds for every vector* $\mathbf{a}_j$ *of matrix* $\mathbf{A}$*, then the solution* $\mathbf{x}_B$ *is optimal.*

If there exists a nonbasic vector $\mathbf{a}_k$ *in matrix* $\mathbf{A}$ *such that* $z_k - c_k = 0$*, and for such vector* $\mathbf{a}_k$ *all coordinates* y_{ik} *are less than or equal to zero,* $i = 1, \ldots, m$*, then there are multiple optimal solutions with unbounded variables.*

Proof. The solutions are computed as shown in Theorem 2.4.3. (see (2.12) on page 47).

As in the cited theorem, the objective value can be computed like this:

$$\hat{z} = z - \theta(z_k - c_k).$$

Since $z_k - c_k = 0$, it follows that $\hat{z} = z$. Therefore, every solution $\overset{\wedge}{\mathbf{x}}$ computed as shown in (2.12) is optimal.   □

2.4.7 The initial basic feasible solution

When solving a linear model in standard form, we will always start with a basic feasible solution. If we choose the canonical basis to start, it will be easy to compute the corresponding basic feasible solution, because $\mathbf{B}^{-1} = \mathbf{B} = \mathbf{I}$. Once the initial basic feasible solution is computed, we apply Theorem 2.4.1 and improve the solution until the optimality conditions stated in Theorem 2.4.2 hold. The initial canonical basis can be of two types.

Case 1. An initial canonical basis formed by slack variables.

Consider the following linear model:

$$\begin{aligned} \max \ z \ &= \mathbf{c}^T\mathbf{x} \\ \text{subject to} \ & \\ \mathbf{A}\mathbf{x} &\leq \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \end{aligned}$$

Assume that $\mathbf{b} \geq \mathbf{0}$. Then, we add vector $\mathbf{y}$ of slack variables and we get the model in standard form.

$$\begin{aligned} \max \ z \ &= \mathbf{c}^T\mathbf{x} + \mathbf{0}^T\mathbf{y} \\ \text{subject to} \ & \\ \mathbf{A}\mathbf{x} + \mathbf{I}\mathbf{y} &= \mathbf{b} \\ \mathbf{x}, \mathbf{y} &\geq \mathbf{0} \end{aligned}$$

Consequently, we can choose basis $\mathbf{B} = \mathbf{I}$. As $\mathbf{B}^{-1} = \mathbf{I}$, it follows that:

- Calculation of the basic solution. It is feasible.

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} = \mathbf{I}\mathbf{b} = \mathbf{b} \geq \mathbf{0}.$$

- Calculation of the objective value. As all vectors in the initial canonical basis $\mathbf{B}$ correspond to slack variables, $\mathbf{c}_B^T = \mathbf{0}$ holds.

$$z = \mathbf{c}_B^T\mathbf{x}_B = \mathbf{0}^T\mathbf{x}_B = 0.$$

- For each vector $\mathbf{a}_j$ of matrix $\mathbf{A}$ we have to compute:

– The coordinate vector.

$$\mathbf{y}_j = \mathbf{B}^{-1}\mathbf{a}_j = \mathbf{I}\mathbf{a}_j = \mathbf{a}_j.$$

– Calculation of the reduced cost coefficients $z_j - c_j$. As all vectors in the initial canonical basis $\mathbf{B}$ correspond to slack variables, $\mathbf{c}_B^T = \mathbf{0}$ holds.

$$z_j - c_j = \mathbf{c}_B^T\mathbf{y}_j - c_j = 0 - c_j = -c_j.$$

As we can see, there is a great advantage in choosing the canonical basis as the initial basis, since then all the calculations needed are directly obtained from the linear model.

Example. Consider the following linear model:

$$\max \quad z \ = \ 2x_1 + 3x_2$$
$$\text{subject to}$$
$$3x_1 + x_2 \leq 2$$
$$x_1 - x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

After adding two slack variables, the model in standard form is the following:

$$\max \quad z \ = \ 2x_1 + 3x_2 + 0x_3 + 0x_4$$
$$\text{subject to}$$
$$3x_1 + x_2 \ + x_3 \qquad\qquad = 2$$
$$x_1 - x_2 \qquad + x_4 \ = 3$$
$$x_1, \ x_2, \ x_3, \ x_4 \geq 0$$

We choose the canonical basis $\mathbf{B} = (\mathbf{a}_3 \ \mathbf{a}_4) = \mathbf{I}$.

- The calculation of the solution. The solution is feasible.

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} = \mathbf{I}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

- The objective value.

$$z = \mathbf{c}_B^T \mathbf{x}_B = (0,0) \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 0$$

- The coordinate vector $\mathbf{y}_j$ and the reduced cost coefficients for each vector of matrix $\mathbf{A}$.

$$\mathbf{a}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \rightarrow \quad \mathbf{y}_1 = \mathbf{B}^{-1}\mathbf{a}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$z_1 - c_1 = \mathbf{c}_B^T \mathbf{y}_1 - c_1 = (0,0) \begin{pmatrix} 3 \\ 1 \end{pmatrix} - 2 = -2$$

$$\mathbf{a}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \rightarrow \quad \mathbf{y}_2 = \mathbf{B}^{-1}\mathbf{a}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$z_2 - c_2 = \mathbf{c}_B^T \mathbf{y}_2 - c_2 = (0,0) \begin{pmatrix} 1 \\ -1 \end{pmatrix} - 3 = -3$$

$$\mathbf{a}_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \rightarrow \quad \mathbf{y}_3 = \mathbf{B}^{-1}\mathbf{a}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$z_3 - c_3 = \mathbf{c}_B^T \mathbf{y}_3 - c_3 = (0,0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 0 = 0$$

$$\mathbf{a}_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \rightarrow \quad \mathbf{y}_4 = \mathbf{B}^{-1}\mathbf{a}_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$z_4 - c_4 = \mathbf{c}_B^T \mathbf{y}_4 - c_4 = (0,0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 0 = 0$$

□

Case 2. Artificial variables in the initial basis.

In the previous case, we assumed that an initial canonical basis is at hand in matrix $\mathbf{A}$ once the model has been written in standard form. However, in many cases such a basis is not readily available. If once the model is in standard form matrix $\mathbf{A}$ has no identity submatrix, then we introduce *artificial variables* to find a starting canonical basis and its corresponding basic feasible solution. We illustrate the use of the artificial variables in the following example.

Example. Consider the following linear model:

$$\begin{aligned}
\max \quad z \quad &= 3x_1 + x_2 \\
\text{subject to} \quad & \\
x_1 + x_2 &\leq 3 \\
x_1 + 2x_2 &\geq 2 \\
x_1, x_2 &\geq 0
\end{aligned}$$

We add the slack variable x_3 to the first constraint and subtract the slack variable x_4 from the second in order to obtain the standard form of the model.

$$\begin{aligned}
\max \quad z \quad &= \quad 3x_1 + x_2 + 0x_3 + 0x_4 \\
\text{subject to} \quad & \\
x_1 + x_2 \quad +x_3 \qquad\quad &= 3 \\
x_1 + 2x_2 \qquad -x_4 \quad &= 2 \\
x_1, \; x_2, \; x_3, \; x_4 &\geq 0
\end{aligned}$$

Matrix $\mathbf{A}$ in the model written in standard form has no identity submatrix. We need to add an artificial variable, $w_1 \geq 0$, to the second constraint in order to get a canonical basis. This leads to the following constraints set:

$$\begin{aligned}
x_1 + x_2 \quad +x_3 \qquad\qquad &= 3 \\
x_1 + 2x_2 \qquad -x_4 + w_1 \; &= 2
\end{aligned}$$

We now choose the canonical basis $\mathbf{B} = (\mathbf{a}_3 \; \mathbf{a}_{w1})$, and compute its corresponding basic feasible solution.

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

However, $\mathbf{x}_B$ is not a solution to the initial linear model, because the artificial variable $w_1 = 2 > 0$; constraint $x_1 + 2x_2 - x_4 = 2$ does not hold. $\qquad\square$

2.4.8 The simplex tableau

In the process of computing the optimal solution for a linear model in standard form, all the calculations that correspond to each of the bases are gathered in a tableau, which is usually called *the simplex tableau*. The process always begins by selecting a canonical basis in matrix $\mathbf{A}$ once the model is in standard form. In case it is not possible to choose a canonical basis with the slack variables, then all the necessary artificial variables are introduced to the model. The simplex tableau is the following:

		Original variables			Auxiliary variables				
		x_1	$\ldots$	x_n	x_{n+1}	$\ldots$	x_j	$\ldots$	
		$z_1 - c_1$	$\ldots$	$z_n - c_n$	$z_{n+1} - c_{n+1}$	$\ldots$	$z_j - c_j$	$\ldots$	z
c_{B1}	$\mathbf{a}_{B1}$	y_{11}	$\ldots$	y_{1n}	$y_{1,n+1}$	$\ldots$	$y_{1,j}$	$\ldots$	x_{B1}
$\vdots$	$\vdots$		$\vdots$			$\vdots$			$\vdots$
c_{Bi}	$\mathbf{a}_{Bi}$	y_{i1}	$\ldots$	y_{in}	$y_{i,n+1}$	$\ldots$	$y_{i,j}$	$\ldots$	x_{Bi}
$\vdots$	$\vdots$		$\vdots$			$\vdots$			$\vdots$
c_{Bm}	$\mathbf{a}_{Bm}$	y_{m1}	$\ldots$	y_{mn}	$y_{m,n+1}$	$\ldots$	$y_{m,j}$	$\ldots$	x_{Bm}

- Original variables of the model, $x_1, \ldots, x_n$, and auxiliary variables, either slack variables or artificial variables, $x_{n+1}, \ldots, x_j, \ldots$, are placed at the top of the tableau.

- Basic vectors, $\mathbf{a}_{B1}, \ldots, \mathbf{a}_{Bi}, \ldots, \mathbf{a}_{Bm}$, are placed at the first column of the tableau.

- Outside of the tableau, to the left of the basic vectors, the basic components of vector $\mathbf{c}^T$ are placed: $c_{B1}, \ldots, c_{Bi}, \ldots, c_{Bm}$.

- The coordinate vectors $\mathbf{y}_j$ for each $\mathbf{a}_j$ relative to the basis are placed in the rest of the columns of the central part of the tableau.

- The components of the basic feasible solution, $x_{B1}, \ldots, x_{Bi}, \ldots, x_{Bm}$, are placed in the last column of the tableau.

- In row zero of the tableau, the reduced cost coefficients $z_j - c_j$ are placed. The objective value z is written in the last column of the row.

Example. Let us consider the linear model in standard form of the example analized on page 54. The simplex tableau corresponding to the canonical basis is the following:

		x_1	x_2	x_3	x_4	
		-2	-3	0	0	0
0	$\mathbf{a}_3$	3	1	1	0	2
0	$\mathbf{a}_4$	1	-1	0	1	3

$\square$

If the initial canonical basis $\mathbf{B}$ is formed by just slack variables, like in the previous example, then the simplex tableau can be written like this:

		$x_1\ x_2\ \ldots\ x_n$	$x_{n+1}\ x_{n+2}\ \ldots x_{n+m}$	
		$\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A} - \mathbf{c}^T$	$\mathbf{c}_B^T\mathbf{B}^{-1}$	$\mathbf{c}_B^T\mathbf{x}_B$
$\mathbf{c}_B$	$\mathbf{B}$	$\mathbf{B}^{-1}\mathbf{A}$	$\mathbf{B}^{-1}$	$\mathbf{x}_B$

In the initial tableau, $\mathbf{B}^{-1} = \mathbf{I}$ holds. For subsequent bases $\mathbf{B}$ and subsequent tableaux, the inverse $\mathbf{B}^{-1}$ is always situated in the columns corresponding to the initial canonical basis. If the initial canonical basis $\mathbf{B}$ is formed by just slack variables, like in the previous example, then $\mathbf{B}^{-1}$ can be found as shown in the previous tableau. But if there are artificial variables in the first canonical basis, then it becomes necessary to identify the columns where $\mathbf{B}^{-1}$ is placed.

2.5 The Big-M method

As we previously said, when matrix $\mathbf{A}$ has no identity submatrix (see example on page 56) we introduce artificial variables to the model in standard form. By proceeding like this, we obtain the initial canonical basis.

However, by introducing artificial variables, we change the problem, since they are not part of the original problem. In order to return to the original problem, we must force artificial variables to zero, because a constraint which has had an artificial variable added is equivalent to the original constraint if and only if the artificial variable is equal to zero.

The Big-M method is used for this purpose. The strategy consists in *penalizing* the artificial variables in the objective function, that is, giving artificial variables an unfavourable status in the objective function. This is done by using a very large positive number M (thus the name of the method), and using a penalty rule which consists in assigning to the artificial variables a cost coefficient of $-M$, in maximization problems.

Example. Consider the following linear model:

$$\max\ z = -5x_1 + 6x_2 + 7x_3$$
$$\text{subject to}$$
$$2x_1 + 10x_2 - 6x_3 \geq 30$$
$$\frac{5}{2}x_1 - 3x_2 + 5x_3 \leq 10$$
$$2x_1 + 2x_2 + 2x_3 = 5$$
$$x_1, x_2, x_3 \geq 0$$

We write the linear model in standard form, add the necessary artificial variables to obtain the initial identity matrix and penalize them in the objective function. This leads to the following:

$$\max\ z = -5x_1 + 6x_2 + 7x_3 + 0x_4 + 0x_5 - Mw_1 - Mw_2$$

subject to

$$
\begin{array}{rrrrrrrl}
2x_1 + 10x_2 - 6x_3 & -x_4 & & +w_1 & & = 30 \\
\frac{5}{2}x_1 - 3x_2 + 5x_3 & & +x_5 & & & = 10 \\
2x_1 + 2x_2 + 2x_3 & & & & +w_2 & = 5 \\
\end{array}
$$

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ w_1,\ w_2 \geq 0$$

Artificial variables w_1 and w_2 are used to obtain the initial canonical basis $\mathbf{B} = \mathbf{I} = (\mathbf{a}_{w_1} \ \mathbf{a}_5 \ \mathbf{a}_{w_2})$. As the objective function has been penalized for the two artificial variables, $\mathbf{c}_B^T = (-M, \ 0, \ -M)$. The penalty affects the reduced cost coefficients $z_j - c_j$ and the objective value z. The initial simplex tableau is:

		x_1	x_2	x_3	x_4	x_5	w_1	w_2	
		$-4M+5$	$-12M-6$	$4M-7$	M	0	0	0	$-35M$
$-M$	$\mathbf{a}_{w1}$	2	10	-6	-1	0	1	0	30
0	$\mathbf{a}_5$	$\frac{5}{2}$	-3	5	0	1	0	0	10
$-M$	$\mathbf{a}_{w2}$	2	2	2	0	0	0	1	5

<div align="right">□</div>

2.6   The simplex algorithm

Consider a linear model in maximization standard form (artificial variables will be added to it if they are needed to obtain the initial canonical basis $\mathbf{B} = \mathbf{I}$, and they will be penalized in the objective function). The simplex algorithm can be summarized in the following steps:

Step 1. Construct the initial simplex tableau.

Step 2.

- If there exists $z_j - c_j < 0$, then the solution may be improved. Go to Step 4.

- If $z_j - c_j \geq 0$ for every vector $\mathbf{a}_j$ of matrix $\mathbf{A}$, then go to Step 3.

Step 3.

- If there exists an artificial variable with a positive value[1], then the problem is **infeasible**. Stop.

---

[1]If there does not exist any artificial variable with positive value, but we can find an artificial variable in the basis which has value zero, then there are two possibilities: either the solution is degenerate or there exist redundant constraints in the model (see examples on page 82.)

- If there is no artificial variable in the basis, then the solution $\mathbf{x}_B$ in the tableau is **optimal**.

  * $z_j - c_j \geq 0$ holds for every vector $\mathbf{a}_j$ of matrix $\mathbf{A}$. If $z_j - c_j > 0$ for every nonbasic vector $\mathbf{a}_j$, then $\mathbf{x}_B$ is the **unique optimal solution**. Stop.

  * $z_j - c_j \geq 0$ holds for every vector $\mathbf{a}_j$ of matrix $\mathbf{A}$. If there exists a nonbasic vector $\mathbf{a}_k$ in matrix $\mathbf{A}$ such that $z_k - c_k = 0$, and at least one of its coordinates y_{ik} is greater than zero, $i = 1, \ldots, m$, then **another basic feasible solution** can be computed. The problem has **multiple optimal solutions**. Go to Step 5.

  * $z_j - c_j \geq 0$ holds for every vector $\mathbf{a}_j$ of matrix $\mathbf{A}$. If there exists a nonbasic vector $\mathbf{a}_k$ in matrix $\mathbf{A}$ such that $z_k - c_k = 0$, and if $y_{ik} \leq 0, i = 1, \ldots, m$ holds for such vector $\mathbf{a}_k$, then the problem has **multiple optimal solutions**, but they are not basic solutions. Stop.

Step 4.

- If there exists a nonbasic vector $\mathbf{a}_j$ in matrix $\mathbf{A}$ such that $z_j - c_j < 0$, and for such vector $\mathbf{a}_j$ there is no positive coordinate in vector $\mathbf{y}_j$, then the solution is **unbounded**. Stop.

- If there exists a nonbasic vector $\mathbf{a}_j$ in matrix $\mathbf{A}$ such that $z_j - c_j < 0$, and for such vector $\mathbf{a}_j$ there exists at least a coordinate greater than zero in vector $\mathbf{y}_j$, then go to Step 5.

Step 5. Select an entering vector $\mathbf{a}_k$ and a leaving vector $\mathbf{a}_r$, according to the following rules:

- Vector $\mathbf{a}_k$ is selected to enter the base, such that:

$$z_k - c_k = \min_j \{z_j - c_j / z_j - c_j \leq 0\}$$

The kth column is called the **pivot column**.

- Vector $\mathbf{a}_r$ is selected to leave the basis, such that:

$$\frac{x_{Br}}{y_{rk}} = \min_i \left\{ \frac{x_{Bi}}{y_{ik}} / y_{ik} > 0 \right\}$$

The rth row is called the **pivot row**.

The coordinate y_{rk} which is both in the pivot column and in the pivot row is called the **pivot element**.

Step 6. Update the tableau.

- In the first column of the tableau, replace the leaving vector in the basis with the entering one.

- The new pivot row is computed by dividing the current pivot row by the pivot element y_{rk}.

- In order to update all other rows, including row zero, we first define the row multipliers.

    * The multiplier for the ith row: $m_i = \dfrac{y_{ik}}{y_{rk}},\ i = 1, \ldots, m,\ i \neq r.$

    * The multiplier for row zero: $m_0 = \dfrac{z_k - c_k}{y_{rk}}.$

    Rows are updated as follows: the new row is equal to the current row $-$ the row multiplier $\times$ the current pivot row.

    It is also possible to use the definition in order to update the reduced cost coefficients $z_j - c_j$ and the objective value z, that is, $z_j - c_j = \mathbf{c}_B^T \mathbf{y}_j - c_j$, $z = \mathbf{c}_B \mathbf{x}_B$.

Once all rows in the new tableau have been updated, go to Step 2.

If the problem has multiple optimal solutions, then new optimal bases will be computed until we reach a basis that had already been found.

2.7   Some additional examples

In this section we solve four linear models by using the simplex algorithm, and interpret the tableau for different kinds of solutions: unique optimal solution, infeasible problem, multiple optimal solutions and unbounded problem.

Example. (A unique optimal solution).

Consider the following linear model.

$$\max \quad z \quad = 6x_1 + 4x_2 + 5x_3 + 5x_4$$

subject to

$$x_1 + x_2 + x_3 + x_4 \leq 3$$
$$2x_1 + x_2 + 4x_3 + x_4 \leq 4$$
$$x_1 + 2x_2 - 2x_3 + 3x_4 \leq 10$$
$$x_1, x_2, x_3, x_4 \geq 0$$

We transform the model into the maximization standard form.

$$\max \quad z = 6x_1 + 4x_2 + 5x_3 + 5x_4 + 0x_5 + 0x_6 + 0x_7$$

subject to

$$x_1 + x_2 + x_3 + x_4 \quad +x_5 \qquad\qquad = 3$$
$$2x_1 + x_2 + 4x_3 + x_4 \qquad +x_6 \qquad = 4$$
$$x_1 + 2x_2 - 2x_3 + 3x_4 \qquad\qquad +x_7 \; = 10$$
$$x_1, \; x_2, \; x_3, \; x_4, \; x_5, \; x_6, \; x_7 \geq 0$$

We construct the initial simplex tableau by considering $\mathbf{B} = (\mathbf{a}_5 \; \mathbf{a}_6 \; \mathbf{a}_7)$ as the initial basis. Since the basis is canonical, and formed by just slack variables, all the initial calculations match the parameters of the model, and thus, they are easily put together in the tableau, as we saw in the example on page 54. The second and third tableaux show the calculations made in the second and third iterations of the simplex algorithm to reach the optimal solution. The pivot element is highlighted by a square, and the row multipliers are out of the tableau, on the right hand side.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7		
		-6	-4	-5	-5	0	0	0	0	
0	$\mathbf{a}_5$	1	1	1	1	1	0	0	3	$m_1 = \frac{y_{11}}{y_{21}} = \frac{1}{2}$
0	$\mathbf{a}_6$	$\boxed{2}$	1	4	1	0	1	0	4	
0	$\mathbf{a}_7$	1	2	-2	3	0	0	1	10	$m_3 = \frac{y_{31}}{y_{21}} = \frac{1}{2}$
		0	-1	7	-2	0	3	0	12	
0	$\mathbf{a}_5$	0	$\frac{1}{2}$	-1	$\boxed{\frac{1}{2}}$	1	$-\frac{1}{2}$	0	1	
6	$\mathbf{a}_1$	1	$\frac{1}{2}$	2	$\frac{1}{2}$	0	$\frac{1}{2}$	0	2	$m_2 = \frac{y_{24}}{y_{14}} = 1$
0	$\mathbf{a}_7$	0	$\frac{3}{2}$	-4	$\frac{5}{2}$	0	$-\frac{1}{2}$	1	8	$m_3 = \frac{y_{34}}{y_{14}} = 5$
		0	1	3	0	4	1	0	16	
5	$\mathbf{a}_4$	0	1	-2	1	2	-1	0	2	
6	$\mathbf{a}_1$	1	0	3	0	-1	1	0	1	
0	$\mathbf{a}_7$	0	-1	1	0	-5	2	1	3	

Since $z_j - c_j \geq 0$ holds for every vector $\mathbf{a}_j$ of matrix $\mathbf{A}$, the iterations of the simplex algorithm are finished. The problem has a **unique optimal solution**.

$$x_1^* = 1, \; x_2^* = 0, \; x_3^* = 0, \; x_4^* = 2, \; x_5^* = 0, \; x_6^* = 0, \; x_7^* = 3, \quad z^* = 16$$

Further details about calculations made in each of the iterations of the simplex algorithm are shown next:

$\boxed{\text{1st iteration}}$

There are negative reduced cost coefficients in the initial tableau, and thus, the solution may be improved.

- The entering vector $\mathbf{a}_k$ is selected. $z_k - c_k = \min_j \{z_j - c_j / z_j - c_j \leq 0\}$.

$$\min\{-6, -4, -5, -5\} = -6 \; \rightarrow \; \mathbf{a}_1 \text{ enters.}$$

The first column of the first tableau is the pivot column.

- The leaving vector $\mathbf{a}_r$ is selected. $\dfrac{x_{Br}}{y_{r1}} = \min\limits_i \left\{ \dfrac{x_{Bi}}{y_{i1}} / y_{i1} > 0 \right\}$.

$$\min\left\{ \frac{3}{1}, \frac{4}{2}, \frac{10}{1} \right\} = \min\{3, 2, 10\} = 2 \;\rightarrow\; \mathbf{a}_6 \;\text{leaves.}$$

The second row of the first tableau is the pivot row.

- Pivot-element: 2

The following calculations are made to get the second tableau, that is, the one corresponding to the basis $\mathbf{B} = (\mathbf{a}_5\; \mathbf{a}_1\; \mathbf{a}_7)$.

- **Pivot row.** In order to compute the second row of the new tableau, we need to divide each value of the pivot row of the first tableau by the pivot element.

$$\frac{1}{2}\,(2,\; 1,\; 4,\; 1,\; 0,\; 1,\; 0,\; 4) = (1, \frac{1}{2}, 2, \frac{1}{2}, 0, \frac{1}{2}, 0, 2)$$

- **1st row.** Multiplier: $m_1 = \frac{y_{11}}{y_{21}} = \frac{1}{2}$. In order to compute the first row of the new tableau, we consider rows in the first tableau and proceed as follows: "first row" $-$ "multiplier" $\times$ "pivot row"

$$(1\,,1\,,1\,,1\,,1\,,0\,,0\,,3) - \frac{1}{2}(2\,,1\,,4\,,1\,,0\,,1\,,0\,,4) =$$

$$= (0\,,\frac{1}{2}\,,-1\,,\frac{1}{2}\,,1\,,-\frac{1}{2}\,,0\,,1)$$

- **3rd row.** Multiplier: $m_3 = \frac{y_{31}}{y_{21}} = \frac{1}{2}$. In order to compute the third row of the new tableau, we consider rows in the first tableau and proceed as follows: "third row" $-$ "multiplier" $\times$ "pivot row"

$$(1\,,2\,,-2\,,3\,,0\,,0\,,1\,,10) - \frac{1}{2}(2\,,1\,,4\,,1\,,0\,,1\,,0\,,4) =$$

$$= (0\,,\frac{3}{2}\,,-4\,,\frac{5}{2}\,,0\,,-\frac{1}{2}\,,1\,,8)$$

- **Row zero.** Multiplier: $\frac{z_1-c_1}{y_{21}} = -\frac{6}{2} = -3$. In order to compute row zero of the new tableau, we consider rows in the first tableau and proceed as follows:

"Row zero" $-$ "multiplier" $\times$ "pivot row"

$$(-6\,,-4\,,-5\,,-5\,,0\,,0\,,0\,,0)-(-3)(2\,,1\,,4\,,1\,,0\,,1\,,0\,,4)=$$

$$=(0\,,-1\,,7\,,-2\,,0\,,3\,,0\,,12)$$

An alternative way to update row zero, reduced cost coefficients $z_j - c_j$, is to use the definition, that is, we compute $z_j - c_j = \mathbf{c}_B^T \mathbf{y}_j - c_j$ by considering the new basis. For instance,

$$z_1 - c_1 = (0\,,\ 6\,,\ 0)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 6 = 6 - 6 = 0$$

- **The objective value.** We use the definition and compute the objective value for the new basis, that is,

$$z = \mathbf{c}_B^T \mathbf{x}_B = (0,\ 6,\ 0)\begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}$$

$\boxed{\text{2. iteration}}$

There are negative reduced cost coefficients in the second tableau, and thus, the solution may be improved. Calculations are made as in the first iteration.

- Entering vector. $\min\{-1, -2\} = -2 \to \mathbf{a}_4$ enters the basis. Pivot column: fourth column of the second tableau.

- Leaving vector. $\min\{\frac{1}{\frac{1}{2}}, \frac{2}{\frac{1}{2}}, \frac{8}{\frac{5}{2}}\} = \min\{2, 4, \frac{16}{5}\} = 2 \to \mathbf{a}_5$ leaves the basis. Pivot row: first row of the second tableau.

- Pivot element: $\dfrac{1}{2}$.

- **Pivot row.**

$$2(0\,,\frac{1}{2}\,,-1\,,\frac{1}{2}\,,1\,,-\frac{1}{2}\,,0\,,1)=(0\,,1\,,-2\,,1\,,2\,,-1\,,0\,,2)$$

- **2nd row.** Multiplier: $m_2 = \dfrac{y_{24}}{y_{14}} = 1$

$$(1\,,\tfrac{1}{2}\,,2\,,\tfrac{1}{2}\,,0\,,\tfrac{1}{2}\,,0\,,2) - 1(0\,,\tfrac{1}{2}\,,-1\,,\tfrac{1}{2}\,,1\,,-\tfrac{1}{2}\,,0\,,1) =$$

$$= (1\,,0\,,3\,,0\,,-1\,,1\,,0\,,1)$$

- **3rd row.** Multiplier: $m_3 = \dfrac{y_{34}}{y_{14}} = 5$

$$(0\,,\tfrac{3}{2}\,,-4\,,\tfrac{5}{2}\,,0\,,-\tfrac{1}{2}\,,1\,,8) - 5(0\,,\tfrac{1}{2}\,,-1\,,\tfrac{1}{2}\,,1\,,-\tfrac{1}{2}\,,0\,,1) =$$

$$= (0\,,-1\,,1\,,0\,,-5\,,2\,,1\,,3)$$

- **Row zero.** Multiplier: $\dfrac{z_4 - c_4}{y_{14}} = -4$

$$(0\,,-1\,,7\,,-2\,,0\,,3\,,0\,,12) + 4(0\,,\tfrac{1}{2}\,,-1\,,\tfrac{1}{2}\,,1\,,-\tfrac{1}{2}\,,0\,,1) =$$

$$= (0\,,1\,,3\,,0\,,4\,,1\,,0\,,16)$$

$\square$

Example. (Infeasible problem). The initial tableau for the following linear model has been constructed in the example of page 59.

$$\max\ z = -5x_1 + 6x_2 + 7x_3 + 0x_4 + 0x_5 - Mw_1 - Mw_2$$

subject to

$$
\begin{aligned}
2x_1 + 10x_2 - 6x_3\ -x_4\quad\quad\ +w_1\quad\quad\quad &= 30 \\
\tfrac{5}{2}x_1 - 3x_2 + 5x_3\quad\quad +x_5\quad\quad\quad\quad &= 10 \\
2x_1 + 2x_2 + 2x_3\quad\quad\quad\quad\quad +w_2\ \ &=\ \ 5 \\
x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ w_1,\ w_2 &\geq 0
\end{aligned}
$$

We apply the simplex algorithm until we reach the optimal tableau. All the calculations can be seen in the following sequence of tableaux:

		x_1	x_2	x_3	x_4	x_5	w_1	w_2		
		$-4M+5$	$-12M-6$	$4M-7$	M	0	0	0	$-35M$	
$-M$	$\mathbf{a}_{w1}$	2	10	-6	-1	0	1	0	30	5
0	$\mathbf{a}_5$	$\frac{5}{2}$	-3	5	0	1	0	0	10	$-\frac{3}{2}$
$-M$	$\mathbf{a}_{w2}$	2	$\boxed{2}$	2	0	0	0	1	5	
		$8M+11$	0	$16M-1$	M	0	0	$6M+3$	$-5M+15$	
$-M$	$\mathbf{a}_{w1}$	-8	0	-16	-1	0	1	-5	5	
0	$\mathbf{a}_5$	$\frac{11}{2}$	0	8	0	1	0	$\frac{3}{2}$	$\frac{35}{2}$	
6	$\mathbf{a}_2$	1	1	1	0	0	0	$\frac{1}{2}$	$\frac{5}{2}$	

$z_j - c_j \geq 0$ holds in the final tableau for every vector $\mathbf{a}_j$ of matrix $\mathbf{A}$, and thus, the simplex algorithm stops. However, the use of the penalty M has not forced the artificial variable w_1 to zero in the final tableau. We conclude that the original model is **infeasible**, because $w_1 = 5$. $\qquad\square$

Example. (Multiple optimal solutions.) Consider the following linear model:

$$\begin{aligned} \min \quad z \ &= 3x_1 + 6x_2 \\ \text{subject to} \quad & \\ x_1 + 2x_2 &\geq 4 \\ x_1 + x_2 &\leq 5 \\ 3x_1 + 4x_2 &\geq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The transformation of the model into the maximization standard form, with the addition of the necessary artificial variables and their penalty in the objective function, yields the following:

$$\max (-z) = -3x_1 - 6x_2 + 0x_3 + 0x_4 + 0x_5 - Mw_1 - Mw_2$$

subject to

$$
\begin{aligned}
x_1 + 2x_2 - x_3 \qquad\qquad + w_1 \qquad &= 4 \\
x_1 + x_2 \qquad + x_4 \qquad\qquad\qquad &= 5 \\
3x_1 + 4x_2 \qquad\qquad - x_5 \qquad + w_2 &= 10 \\
x_1,\, x_2,\, x_3,\, x_4, x_5,\, w_1,\, w_2 &\geq 0
\end{aligned}
$$

The simplex algorithm tableaux are the following:

		x_1	x_2	x_3	x_4	x_5	w_1	w_2		
		$-4M+3$	$-6M+6$	M	0	M	0	0	$-14M$	
$-M$	$\mathbf{a}_{w1}$	1	$\boxed{2}$	-1	0	0	1	0	4	
0	$\mathbf{a}_4$	1	1	0	1	0	0	0	5	$\frac{1}{2}$
$-M$	$\mathbf{a}_{w2}$	3	4	0	0	-1	0	1	10	2
		$-M$	0	$-2M+3$	0	M	$3M-3$	0	$-2M-12$	
-6	$\mathbf{a}_2$	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	0	2	$-\frac{1}{4}$
0	$\mathbf{a}_4$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	3	$\frac{1}{4}$
$-M$	$\mathbf{a}_{w2}$	1	0	$\boxed{2}$	0	-1	-2	1	2	
		$-\frac{3}{2}$	0	0	0	$\frac{3}{2}$	M	$M-\frac{3}{2}$	-15	
-6	$\mathbf{a}_2$	$\frac{3}{4}$	1	0	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{5}{2}$	$\frac{3}{2}$
0	$\mathbf{a}_4$	$\frac{1}{4}$	0	0	1	$\frac{1}{4}$	0	$-\frac{1}{4}$	$\frac{5}{2}$	$\frac{1}{2}$
0	$\mathbf{a}_3$	$\boxed{\frac{1}{2}}$	0	1	0	$-\frac{1}{2}$	-1	$\frac{1}{2}$	1	
		0	0	3	0	0	$M-3$	M	-12	
-6	$\mathbf{a}_2$	0	1	$-\frac{3}{2}$	0	$\boxed{\frac{1}{2}}$	$\frac{3}{2}$	$-\frac{1}{2}$	1	
0	$\mathbf{a}_4$	0	0	$-\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	2	1
-3	$\mathbf{a}_1$	1	0	2	0	-1	-2	1	2	-2

$z_j - c_j \geq 0$ holds in the final tableau for every vector of the matrix. Moreover, there is no artificial variable in the optimal basis, and thus, we have an optimal

basic feasible solution for the original linear model: $x_1^* = 2$, $x_2^* = 1$, $x_3^* = 0$, $x_4^* = 2$, $x_5^* = 0$, $z^* = 12$. This solution corresponds to the extreme point A of the graphic.

If we look at the optimal tableau, we see that $z_5 - c_5 = 0$ holds for the nonbasic vector $\mathbf{a}_5$. This means that the problem has **multiple optimal solutions**; it is possible to compute another optimal basic feasible solution. To do so, we select vector $\mathbf{a}_5$ to enter the basis, and using the leaving vector rule, we choose vector $\mathbf{a}_2$ as the leaving vector. Given that the pivot element is $\frac{1}{2}$, the new tableau is:

		x_1	x_2	x_3	x_4	x_5	w_1	w_2	
		0	0	3	0	0	$M-3$	M	-12
0	$\mathbf{a}_5$	0	2	-3	0	1	3	-1	2
0	$\mathbf{a}_4$	0	-1	1	1	0	-1	0	1
-3	$\mathbf{a}_1$	1	2	-1	0	0	1	0	4

Another optimal basic feasible solution: $x_1^* = 4$, $x_2^* = 0$, $x_3^* = 0$, $x_4^* = 1$, $x_5^* = 2$, $z^* = 12$. This solution corresponds to the extreme point B of the graphical solution on page 71.

In the second optimal tableau we see that once again the condition for multiple optimal solutions holds, because $z_2 - c_2 = 0$, being vector $\mathbf{a}_2$ nonbasic. However, if we enter vector $\mathbf{a}_2$ to the basis and select $\mathbf{a}_5$ as the leaving vector, this yields the optimal solution previously computed. Thus, the simplex algorithm stops.

All points in the segment between the extreme points A and B are also optimal solutions to the problem.

$\square$
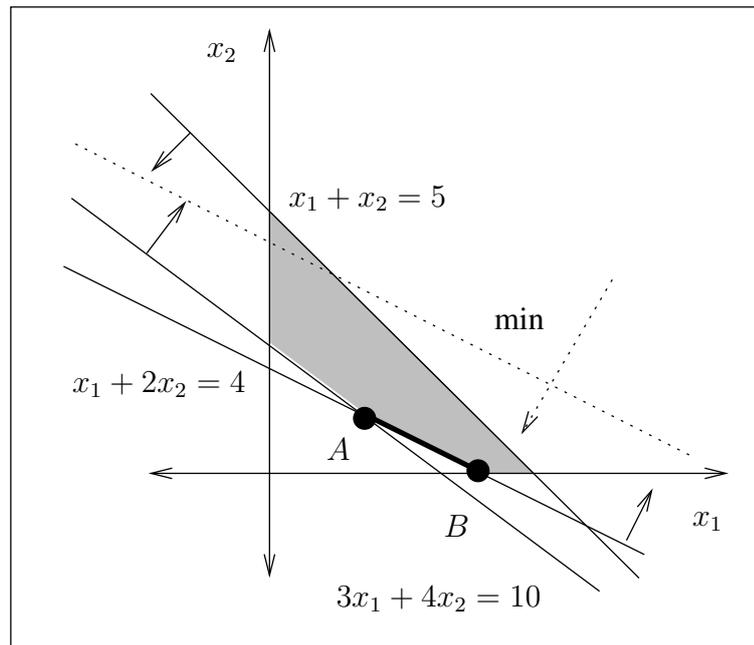
Example. (Unbounded solution.)
Consider the following linear model:

$$\max \ z \ = \ x_1 - 3x_2$$
$$\text{subject to}$$
$$2x_1 + 2x_2 \geq 4$$
$$-4x_1 - 2x_2 \leq -6$$
$$x_1, x_2 \geq 0$$

After transforming it to the standard form and adding the needed artificial variables to obtain the initial $\mathbf{B} = \mathbf{I}$ canonical basis, the model can be written as:

$$\max \ z = x_1 - 3x_2 + 0x_3 + 0x_4 - Mw_1 - Mw_2$$

$$\text{subject to}$$

$$2x_1 + 2x_2 \ -x_3 \qquad +w_1 \qquad = 4$$
$$4x_1 + 2x_2 \qquad -x_4 \qquad +w_2 \ = 6$$
$$x_1, \ x_2, \ x_3, \ x_4, \ w_1, \ w_2 \geq 0$$

The following sequence of tableaux show the calculations made along the three iterations of the simplex algorithm.

		x_1	x_2	x_3	x_4	w_1	w_2		
		$-6M-1$	$-4M+3$	M	M	0	0	$-10M$	
$-M$	$\mathbf{a}_{w1}$	2	2	-1	0	1	0	4	$\frac{1}{2}$
$-M$	$\mathbf{a}_{w2}$	$\boxed{4}$	2	0	-1	0	1	6	
		0	$-M+\frac{7}{2}$	M	$-\frac{1}{2}M-\frac{1}{4}$	0	$\frac{3}{2}M+\frac{1}{4}$	$-M+\frac{3}{2}$	
$-M$	$\mathbf{a}_{w1}$	0	$\boxed{1}$	-1	$\frac{1}{2}$	1	$-\frac{1}{2}$	1	
1	$\mathbf{a}_1$	1	$\frac{1}{2}$	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{1}{2}$
		0	0	$\frac{7}{2}$	-2	$M-\frac{7}{2}$	$M+2$	-2	
-3	$\mathbf{a}_2$	0	1	-1	$\boxed{\frac{1}{2}}$	1	$-\frac{1}{2}$	1	
1	$\mathbf{a}_1$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1
		0	4	$-\frac{1}{2}$	0	$M+\frac{1}{2}$	M	2	
0	$\mathbf{a}_4$	0	2	-2	1	2	-1	2	
1	$\mathbf{a}_1$	1	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	2	

In the last tableau we can see that vector $\mathbf{a}_3$ is nonbasic, that $z_3 - c_3 < 0$ holds and that $y_{i3} \leq 0$, $i = 1, 2$. It follows that the solution is unbounded.
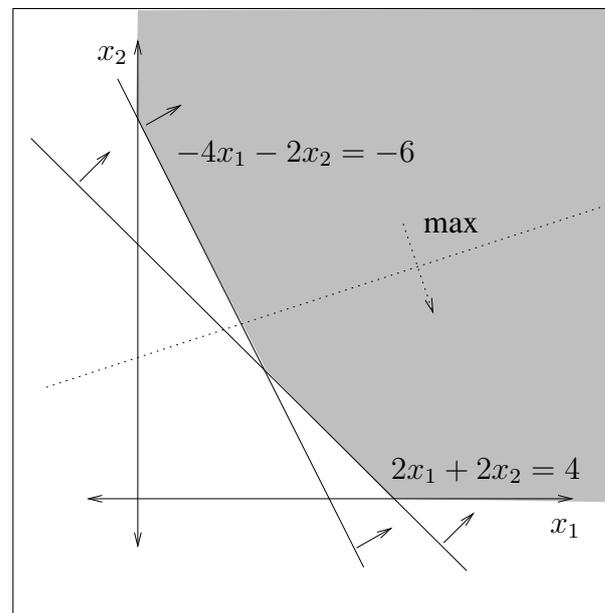
If we look at the graphical solution, we can see that moving the objective function line in the optimization direction (bottom right direction), it will always intersect the feasible region. Therefore, z has an arbitrarily large value. The solution is unbounded. □

2.8 The two-phase method

The two-phase method is analog to the Big-M method in the sense that both are used when artificial variables are needed to get the initial $\mathbf{B} = \mathbf{I}$ canonical basis in a model written in standard form. The aim of the two methods is to reach an optimal solution where all the artificial variables are zero. To accomplish it, the Big-M method uses the strategy of penalizing them in the objective function, while the two-phase method minimizes the sum of the artificial variables in a first phase.

The two-phase method is outlined here:

Phase 1. In the first phase, the constraints of the linear model are considered, but instead of optimizing the original objective function, a new objective function is defined: to minimize the sum of the artificial variables. The simplex algorithm is used to optimize the model, and the following two cases may arise:

- If the optimal objective value is greater than zero, then the original linear model is infeasible.

- Otherwise, the original linear model is feasible. Go to Phase 2.

Phase 2. In the second phase, the original objective function is optimized, taking the optimal tableau obtained in the first phase as starting point. The artificial variables are discarded, and as the objective function of the model has been changed, the reduced cost coefficients $z_j - c_j$ must be updated. Afterwards, the simplex algorithm is used until the optimality condition holds.

Next, two linear models are solved by using the two-phase method and details are discussed.

Example. Consider the following linear model.

$$\max \quad z \ = 2x_1 + 3x_2 - 5x_3$$
$$\text{subject to}$$
$$2x_1 + 2x_2 + 2x_3 = 14$$
$$-2x_1 + 5x_2 - x_3 \leq -10$$
$$x_1, x_2, x_3 \geq 0$$

We transform the model into the standard form and introduce the artificial variables w_1 and w_2:

$$\max \quad z \ = 2x_1 + 3x_2 - 5x_3 + 0x_4 + 0w_1 + 0w_2$$
$$\text{subject to}$$
$$2x_1 + 2x_2 + 2x_3 \qquad +w_1 \qquad = 14$$
$$2x_1 - 5x_2 + x_3 \ -x_4 \qquad +w_2 \ = 10$$
$$x_1, \ x_2, \ x_3, \ x_4, \ w_1, \ w_2 \geq 0$$

Phase 1. The phase 1 objective is to minimize the sum of the two artificial variables introduced in the model, that is, $\min \ z' \ = w_1 + w_2$. We transform it into the maximization form before using the simplex algorithm.

$$\max (-z') = 0x_1 + 0x_2 + 0x_3 + 0x_4 - w_1 - w_2$$

subject to

$$2x_1 + 2x_2 + 2x_3 \qquad +w_1 \qquad = 14$$
$$2x_1 - 5x_2 + x_3 \ -x_4 \qquad +w_2 \ = 10$$
$$x_1, \ x_2, \ x_3, \ x_4, \ w_1, \ w_2 \geq 0$$

The simplex algorithm tableaux are the following:

		x_1	x_2	x_3	x_4	w_1	w_2		
		-4	3	-3	1	0	0	-24	
-1	$\mathbf{a}_{w1}$	2	2	2	0	1	0	14	1
-1	$\mathbf{a}_{w2}$	$\boxed{2}$	-5	1	-1	0	1	10	
		0	-7	-1	-1	0	2	-4	
-1	$\mathbf{a}_{w1}$	0	$\boxed{7}$	1	1	1	-1	4	
0	$\mathbf{a}_1$	1	$-\frac{5}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	5	$-\frac{5}{14}$
		0	0	0	0	1	1	0	
0	$\mathbf{a}_2$	0	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{4}{7}$	
0	$\mathbf{a}_1$	1	0	$\frac{6}{7}$	$-\frac{1}{7}$	$\frac{5}{14}$	$\frac{1}{7}$	$\frac{45}{7}$	

Since $z_j - c_j \geq 0$ holds for every vector of matrix $\mathbf{A}$, the optimal tableau for phase 1 has been found. Moreover, since $z'^* = 0$, the original linear model is feasible, and we proceed to compute the optimal solution in phase 2.

Phase 2. We optimize the objective function of the original linear model: max $z = 2x_1 + 3x_2 - 5x_3$. We update the reduced cost coefficients and z in the optimal tableau previously obtained:

- $z_1 - c_1 = \mathbf{c}_B^T \mathbf{y}_1 - c_1 = (3 \ , \ 2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - 2 = 0.$

- $z_2 - c_2 = \mathbf{c}_B^T \mathbf{y}_2 - c_2 = (3 \ , \ 2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 3 = 0.$

- $z_3 - c_3 = \mathbf{c}_B^T \mathbf{y}_3 - c_3 = (3 \,,\, 2) \begin{pmatrix} \frac{1}{7} \\ \frac{6}{7} \end{pmatrix} + 5 = \frac{50}{7}.$

- $z_4 - c_4 = \mathbf{c}_B^T \mathbf{y}_4 - c_4 = (3 \,,\, 2) \begin{pmatrix} \frac{1}{7} \\ -\frac{1}{7} \end{pmatrix} - 0 = \frac{1}{7}.$

- $z = \mathbf{c}_B^T \mathbf{x}_B = (3 \,,\, 2) \begin{pmatrix} \frac{4}{7} \\ \frac{45}{7} \end{pmatrix} = \frac{102}{7}.$

Since the artificial variables are not included in the objective function considered in phase 2, we eliminate the corresponding columns. After computing the correct reduced cost coefficients, we obtain the initial tableau for phase 2:

		x_1	x_2	x_3	x_4	
		0	0	$\frac{50}{7}$	$\frac{1}{7}$	$\frac{102}{7}$
3	$\mathbf{a}_2$	0	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{4}{7}$
2	$\mathbf{a}_1$	1	0	$\frac{6}{7}$	$-\frac{1}{7}$	$\frac{45}{7}$

Since $z_j - c_j \geq 0$ holds for every vector of matrix $\mathbf{A}$, the tableau displays the optimal solution for the original linear model.

$$x_1^* = \frac{45}{7}, \quad x_2^* = \frac{4}{7}, \quad x_3^* = 0, \quad x_4^* = 0, \quad z^* = \frac{102}{7}.$$

$\square$

Example. We now solve the linear model of the example on page 71 by using the two phase method.

$$\max \ z = x_1 - 3x_2 + 0x_3 + 0x_4 + 0w_1 + 0w_2$$

subject to

$$2x_1 + 2x_2 \ -x_3 \qquad\qquad +w_1 \qquad\quad = 4$$
$$4x_1 + 2x_2 \qquad\quad -x_4 \qquad\qquad +w_2 \ = 6$$
$$x_1, \ x_2, \ x_3, \ x_4, \ w_1, \ w_2 \geq 0$$

Phase 1. Objective function: $\min z' = w_1 + w_2 \rightarrow \max(-z') = -w_1 - w_2$.

$$\max(-z') = 0x_1 + 0x_2 + 0x_3 + 0x_4 - w_1 - w_2.$$

		x_1	x_2	x_3	x_4	w_1	w_2		
		-6	-4	1	1	0	0	-10	
-1	$\mathbf{a}_{w1}$	2	2	-1	0	1	0	4	$\frac{1}{2}$
-1	$\mathbf{a}_{w2}$	$\boxed{4}$	2	0	-1	0	1	6	
		0	-1	1	$-\frac{1}{2}$	0	$\frac{3}{2}$	-1	
-1	$\mathbf{a}_{w1}$	0	$\boxed{1}$	-1	$\frac{1}{2}$	1	$-\frac{1}{2}$	1	
0	$\mathbf{a}_{1}$	1	$\frac{1}{2}$	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{3}{2}$	$\frac{1}{2}$
		0	0	0	0	1	1	0	
0	$\mathbf{a}_{2}$	0	1	-1	$\frac{1}{2}$	1	$-\frac{1}{2}$	1	
0	$\mathbf{a}_{1}$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	1	

Phase 2. The objective function of the original linear model: $\max z = x_1 - 3x_2$. We eliminate the two columns which correspond to the artificial variables, and update the reduced cost coefficients and the objective value:

		x_1	x_2	x_3	x_4		
		0	0	$\frac{7}{2}$	-2	-2	
-3	$\mathbf{a}_{2}$	0	1	-1	$\boxed{\frac{1}{2}}$	1	
1	$\mathbf{a}_{1}$	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	1	-1
		0	4	$-\frac{1}{2}$	0	2	
0	$\mathbf{a}_{4}$	0	2	-2	1	2	
1	$\mathbf{a}_{1}$	1	1	$-\frac{1}{2}$	0	2	

$z_3 - c_3 < 0$ holds in the final tableau, being all the components of vector $\mathbf{y}_3$ negative. Therefore, the problem is unbounded. $\square$

2.9   The revised simplex method

The simplex algorithm described in the previous sections can be executed using a smaller amount of calculations. The *revised simplex method* is a systematic procedure to implement the steps of the simplex method in a way that saves storage space.

Let us consider a linear model in standard form:

$$\max \ z \ = \mathbf{c}^T\mathbf{x}$$
$$\text{subject to}$$
$$\mathbf{A}\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

Suppose that we are given a basis $\mathbf{B}$ and a basic feasible solution $\mathbf{x}_B$. For each nonbasic vector, we need to compute the reduced cost coefficients $z_j - c_j = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{a}_j - c_j$ and see if the actual basic feasible solution can be improved. If there exists at least one negative $z_j - c_j$, then vector $\mathbf{a}_k$ is selected to enter the basis if it satisfies the entering rule:

$$z_k - c_k \ = \ \min_j \left\{ \ z_j - c_j \ / \ z_j - c_j \leq 0 \ \right\}.$$

In order to select the leaving vector, we need to compute its coordinate vector, that is, the pivot column $\mathbf{y}_k = \mathbf{B}^{-1}\mathbf{a}_k$. By using those coordinates and the components in vector $\mathbf{x}_B$, vector $\mathbf{a}_r$ is selected to leave the basis if it satisfies the leaving rule:

$$\frac{x_{Br}}{y_{rk}} \ = \ \min_i \left\{ \ \frac{x_{Bi}}{y_{ik}} \ / \ y_{ik} > 0 \ \right\}.$$

In order to make all the necessary calculations we need to know the inverse basis $\mathbf{B}^{-1}$, which changes each time the basis is changed. All we need then are known parameters, present in the linear model. Thus, we just need a reduced tableau, called the *revised simplex tableau*.

		$x_{n+1} \ x_{n+2} \ \ldots x_{n+m}$	
		$\mathbf{c}_B^T\mathbf{B}^{-1}$	$\mathbf{c}_B^T\mathbf{x}_B$
$\mathbf{c}_B$	$\mathbf{B}$	$\mathbf{B}^{-1}$	$\mathbf{x}_B$

Example. Let us consider the linear model of the example on page 62.

$$\max \ z = 6x_1 + 4x_2 + 5x_3 + 5x_4 + 0x_5 + 0x_6 + 0x_7$$

subject to

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 \ + x_5 \qquad\qquad\qquad &= 3 \\
2x_1 + x_2 + 4x_3 + x_4 \qquad +x_6 \qquad\quad &= 4 \\
x_1 + 2x_2 - 2x_3 + 3x_4 \qquad\qquad +x_7 &= 10 \\
x_1, \ x_2, \ x_3, \ x_4, \ x_5, \ x_6, \ x_7 &\geq 0
\end{aligned}
$$

We choose the canonical basis $\mathbf{B} = (\mathbf{a}_5 \ \mathbf{a}_6 \ \mathbf{a}_7)$ and make the needed calculation to construct the corresponding initial tableau.

$$
\mathbf{B} =
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \qquad
\mathbf{N} =
\begin{pmatrix}
1 & 1 & 1 & 1 \\
2 & 1 & 4 & 1 \\
1 & 2 & -2 & 3
\end{pmatrix}
$$

$$
\mathbf{c}_B^T = (0 \ , \ 0 \ , \ 0), \qquad \mathbf{c}_B^T \mathbf{B}^{-1} = (0, 0, 0), \qquad \mathbf{c}_N^T = (6 \ , \ 4 \ , \ 5 \ , \ 5)
$$

$$
\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} =
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
3 \\
4 \\
10
\end{pmatrix}
=
\begin{pmatrix}
3 \\
4 \\
10
\end{pmatrix}
$$

The initial revised simplex tableau is the following:

		x_5	x_6	x_7	
		0	0	0	0
0	$\mathbf{a}_5$	1	0	0	3
0	$\mathbf{a}_6$	0	1	0	4
0	$\mathbf{a}_7$	0	0	1	10

In order to determine whether the solution may be improved, we need to compute the $z_j - c_j$ for the nonbasic vectors $\mathbf{N} = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4)$.

$$\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N} - \mathbf{c}_N^T = (0,0,0) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 1 \\ 1 & 2 & -2 & 3 \end{pmatrix} - (6,4,5,5) = (-6,-4,-5,-5).$$

$$z_1 - c_1 = \min\{-6,-4,-5,-5\} = -6 \quad \to \quad \mathbf{a}_1 \text{ enters the basis.}$$

Let us compute the pivot column $\mathbf{y}_1$.

$$\mathbf{y}_1 = \mathbf{B}^{-1}\mathbf{a}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Vector $\mathbf{a}_r$ is selected to leave the basis according to the leaving vector rule:

$$\frac{x_{Br}}{y_{r1}} = \min\left\{\frac{3}{1}, \frac{4}{2}, \frac{10}{1}\right\} = 2 \quad \to \quad \mathbf{a}_6 \text{ leaves the basis.}$$

The second row of the tableau is the pivot row, and the pivot element is 2. The multiplier for the first row is $\frac{1}{2}$, for the third is $\frac{1}{2}$, and for row zero is $-\frac{6}{2}$. The new tableau is the following:

		x_5	x_6	x_7	
		0	3	0	12
0	$\mathbf{a}_5$	1	$-\frac{1}{2}$	0	1
6	$\mathbf{a}_1$	0	$\frac{1}{2}$	0	2
0	$\mathbf{a}_7$	0	$-\frac{1}{2}$	1	8

Now, the basis is $\mathbf{B} = (\mathbf{a}_5\ \mathbf{a}_1\ \mathbf{a}_7)$. We proceed to compute the reduced cost coefficients for nonbasic vectors $\mathbf{N} = (\mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_6)$.

$$\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N} - \mathbf{c}_N^T = (0, 3, 0) \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 4 & 1 & 1 \\ 2 & -2 & 3 & 0 \end{pmatrix} - (4, 5, 5, 0) = (-1, 7, -2, 3)$$

$$z_4 - c_4 = \min\{-1, -2\} = -2 \quad \rightarrow \quad \mathbf{a}_4 \text{ enters the basis.}$$

We compute the pivot column.

$$\mathbf{y}_4 = \mathbf{B}^{-1} \mathbf{a}_4 = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{5}{2} \end{pmatrix}$$

Vector $\mathbf{a}_r$ is selected to leave the basis according to the leaving vector rule:

$$\frac{x_{Br}}{y_{r4}} = \min\left\{ \frac{1}{\frac{1}{2}}, \frac{2}{\frac{1}{2}}, \frac{8}{\frac{5}{2}} \right\} = \min\left\{ 2, 4, \frac{16}{5} \right\} = 2 \quad \rightarrow \quad \mathbf{a}_5 \text{ leaves the basis.}$$

The first row of the tableau is the pivot row, and the pivot element is $\frac{1}{2}$. The multiplier for the second row is 1, for the third row is 5 and for row zero is -4. We update the tableau for the new basis and obtain:

		x_5	x_6	x_7	
		4	1	0	16
5	$\mathbf{a}_4$	2	−1	0	2
6	$\mathbf{a}_1$	−1	1	0	1
0	$\mathbf{a}_7$	−5	2	1	3

Now, the basis is $\mathbf{B} = (\mathbf{a}_4 \ \mathbf{a}_1 \ \mathbf{a}_7)$. We proceed to compute the reduced cost coefficients for nonbasic vectors $\mathbf{N} = (\mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{a}_5 \ \mathbf{a}_6)$.

$$\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N} - \mathbf{c}_N^T = (4, 1, 0) \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 4 & 0 & 1 \\ 2 & -2 & 0 & 0 \end{pmatrix} - (4, 5, 0, 0) = (1, 3, 4, 1).$$

Since there is no negative $z_j - c_j$, the solution is optimal. Moreover, it is unique: $x_1^* = 1$, $x_2^* = 0$, $x_3^* = 0$, $x_4^* = 2$, $x_5^* = 0$, $x_6^* = 0$, $x_7^* = 3$, $z^* = 16$.   □

2.10   Some remarks

1. Round-off errors. As the simplex algorithm progresses from one basis to another, the inevitable small round-off errors introduced at each stage accumulate. This happens because computers can carry only a limited number of decimal places of accuracy, and so, computations involving real numbers are approximate. Even though such approximations can be very accurate, the errors tend to accumulate. In practice, error propagation does not usually invalidate results obtained by the simplex algorithm. However, if the error becomes significant to the point where the current stored value $\mathbf{B}^{-1}$ is highly inaccurate, the optimal basic feasible solution obtained may not satisfy the constraints, or if it does, it may not be the optimal solution. This is detected by directly evaluating the error $\mathbf{B}\mathbf{x}_B - \mathbf{b}$ periodically. If $\mathbf{B}\mathbf{x}_B \neq \mathbf{b}$ and the error is considered to be significant, then it is advisable to directly invert the current basis $\mathbf{B}$ as accurately as possible, and proceed from that point with the iterations of the algorithm.

2. Artificial variables in the optimal basis. In some cases, once the simplex algorithm has reached the optimal tableau, and even though we may force the artificial variables to zero, one or more of these variables may still be basic. This fact can show two different situations: either there are redundant constraints in the model, or the solution is degenerate. Let us illustrate them with two examples.

Example. (Redundant constraints). Consider the following linear model:

$$\begin{aligned}
\max \quad z \quad &= x_1 + 2x_2 - x_3 \\
\text{subject to} \quad & \\
2x_1 - x_2 + x_3 &= 12 \\
-x_1 + 2x_2 + x_3 &= 10 \\
x_1 + x_2 + 2x_3 &= 22 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}$$

We introduce artificial variables w_1, w_2 and w_3, penalize the objective function and use the simplex algorithm. After three iterations of the algorithm, we get the optimal tableau.

		x_1	x_2	x_3	w_1	w_2	w_3		
		$-2M-1$	$-2M-2$	$-4M+1$	0	0	0	$-44M$	
$-M$	$\mathbf{a}_{w1}$	2	-1	1	1	0	0	12	1
$-M$	$\mathbf{a}_{w2}$	-1	2	$\boxed{1}$	0	1	0	10	
$-M$	$\mathbf{a}_{w3}$	1	1	2	0	0	1	22	2
		$-6M$	$6M-4$	0	0	$4M-1$	0	$-4M-10$	
$-M$	$\mathbf{a}_{w1}$	$\boxed{3}$	-3	0	1	-1	0	2	
-1	$\mathbf{a}_3$	-1	2	1	0	1	0	10	$-\frac{1}{3}$
$-M$	$\mathbf{a}_{w3}$	3	-3	0	0	-2	1	2	1
		0	-4	0	$2M$	$2M-1$	0	-10	
1	$\mathbf{a}_1$	1	-1	0	$\frac{1}{3}$	$-\frac{1}{3}$	0	$\frac{2}{3}$	-1
-1	$\mathbf{a}_3$	0	$\boxed{1}$	1	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{32}{3}$	
$-M$	$\mathbf{a}_{w3}$	0	0	0	-1	-1	1	0	0
		0	0	4	$2M+\frac{4}{3}$	$2M+\frac{5}{3}$	0	$\frac{98}{3}$	
1	$\mathbf{a}_1$	1	0	1	$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{34}{3}$	
2	$\mathbf{a}_2$	0	1	1	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{32}{3}$	
$-M$	$\mathbf{a}_{w3}$	0	0	0	-1	-1	1	0	

The artificial vector $\mathbf{a}_{w3}$ is in the optimal basis even though we have driven it to zero. In this particular case, there is a redundant constraint in the model. If we analyze the constraints, we can verify that the third constraint can be obtained by adding up the first and the second constraints. Thus, the third constraint is redundant and should be omitted from the model.

$\square$

Example. (Degenerate solution). Consider the following linear model:

$$\max \quad z \; = x_1 + x_2 + 3x_3$$

subject to

$$x_1 + 5x_2 + x_3 \leq 7$$
$$x_1 - x_2 + x_3 = 5$$
$$\frac{1}{2}x_1 - 2x_2 + x_3 = 5$$
$$x_1, x_2, x_3 \geq 0$$

We add the slack variable x_4 and the artificial variables w_1 and w_2 to the model, penalize the objective function and proceed with the simplex algorithm; the optimal tableau is found in one iteration.

		x_1	x_2	x_3	x_4	w_1	w_2	
		$-\frac{3}{2}M - 1$	$3M - 1$	$-2M - 3$	0	0	0	$-10M$
0	$\mathbf{a}_4$	1	5	1	1	0	0	7
$-M$	$\mathbf{a}_{w1}$	1	-1	$\boxed{1}$	0	1	0	5
$-M$	$\mathbf{a}_{w2}$	$\frac{1}{2}$	-2	1	0	0	1	5
		$\frac{1}{2}M + 2$	$M - 4$	0	0	$2M + 3$	0	15
0	$\mathbf{a}_4$	0	6	0	1	-1	0	2
3	$\mathbf{a}_3$	1	-1	1	0	1	0	5
$-M$	$\mathbf{a}_{w2}$	$-\frac{1}{2}$	-1	0	0	-1	1	0

As we can see, there is an artificial vector in the optimal basis, even though it has been driven to zero, as in the previous example. However, in this case there is no redundant constraint in the model.

In this example, it is possible to compute a new optimal tableau in which the artificial vector is nonbasic. The optimal solution to the problem is degenerate.

		x_1	x_2	x_3	x_4	w_1	w_2	
		4	0	0	0	$7+M$	$-4+M$	15
0	$\mathbf{a}_4$	-3	0	0	1	-7	6	2
3	$\mathbf{a}_3$	$\frac{3}{2}$	0	1	0	2	-1	5
1	$\mathbf{a}_2$	$\frac{1}{2}$	1	0	0	1	-1	0

$\square$

3. Cycling. Normally, the simplex algorithm stops after a finite number of iterations. However, if the solution is degenerate, it may cause the algorithm to cycle and iterate indefinitely without ever reaching termination but simply switching from one basis to another[2]. This may happen if ties at the leaving vector rule are not correctly broken.

Some rules have been specified in order to prevent cycling, and thus guarantee finite convergence of the simplex algorithm. They are designed to ensure that none of the previous bases visited by the simplex algorithm are repeated. As there is a finite number of bases, this automatically guarantees stopping after a finite number of iterations. Some cycling prevention rules are the *lexicographic rule* for selecting the leaving vector and *Bland's rule* for selecting the entering and the leaving vectors.

4. Computational complexity of the simplex algorithm. The simplex algorithm is of exponential complexity. However, the performance of the simplex algorithm is usually appropriate; it has been empirically proved to reach the optimal solution quite efficiently. In most of the cases, the algorithm is observed to take $\frac{3m}{2}$ iterations, being $\mathbf{A}$ an $m \times n$ matrix. This means that the convergence of the algorithm is more sensitive to the number of constraints than to the number of variables of the model.

Several attempts have been carried out by researchers to find a polynomial-time algorithm. Karmakar's algorithm is one of them.

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[2]An example can be seen in *Linear programming and network flows*. M. S. Bazaraa et al.