Chapter 1

Linear Modeling and Graphical Solution

Linear programming is an important branch of Operations Research. This mathematical technique consists of a set of methods used to obtain the best solution to linear optimization problems subject to constraints, such as practical contexts when the optimal distribution of limited resources must be found. There is a wide variety of problems which can be represented by a linear programming model. Some examples are the problem of assigning resources to tasks, production planning, transportation of goods, product-mix problems, etc.

In linear programming, a mathematical model is used to describe the problem. The adjective *linear* means that all functions used to define the model are linear.

1.1 The linear model

A linear model deals with optimizing (maximizing or minimizing) a linear function with several variables, given certain linear constraint inequalities.

$$\text{opt} \;\; z = \mathbf{c}^T\mathbf{x} \tag{1.1}$$

subject to

$$\mathbf{A}\mathbf{x} \overset{\leq}{\geq} \mathbf{b} \tag{1.2}$$

$$\mathbf{x} \geq \mathbf{0} \tag{1.3}$$

Here, (1.1) is the linear function to be optimized, and it is called the *objective function*. The inequalities (1.2) are the *constraints* that variables must hold and (1.3) are the *nonnegativity constraints*.

The elements that appear in the model are the following:

- $\mathbf{x}$, is the vector of *decision variables*, and contains n variables.
- $\mathbf{c}^T$, is the vector of *cost coefficients*, and contains n constants.
- $\mathbf{b}$, is the *right-hand-side vector*, with m constants.
- Matrix $\mathbf{A}$, with m rows and n columns is called the *constraint matrix*. Each coefficient a_{ij} in $\mathbf{A}$ represents the amount of resource i, $i = 1, \ldots, m$, needed to perform a unit of activity j, $j = 1, \ldots, n$, and are called the *technological coefficients*.

Vectors $\mathbf{c}^T$ and $\mathbf{b}$ and matrix $\mathbf{A}$ are known parameters in the linear model; vector $\mathbf{x}$ contains the variables whose values have to be determined in order to find the optimal way to assign resources to activities.

1.2 Notation

The linear model can be stated using different notations.

1. We can write the linear model in the following way:

$$
\begin{aligned}
\text{opt } z &= c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} & \\
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\overset{\leq}{\underset{\geq}{=}} b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\overset{\leq}{\underset{\geq}{=}} b_2 \\
\vdots \qquad \vdots \qquad \ddots \qquad \vdots &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\overset{\leq}{\underset{\geq}{=}} b_m \\
x_1, x_2, \ldots, x_n &\geq 0
\end{aligned}
$$

2. Matrix notation:

$$\text{opt } z = (c_1, \ldots, c_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

subject to

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{matrix} \le \\ = \\ \ge \end{matrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$(x_1, x_2, \cdots, x_n)^T \ge (0, 0, \cdots, 0)^T$$

3. Denoting by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ the n columns of matrix $\mathbf{A}$, the linear model can be represented as follows:

$$\text{opt } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to

$$\mathbf{a}_1x_1 + \mathbf{a}_2x_2 + \cdots + \mathbf{a}_nx_n \overset{\le}{\gtreqless} \mathbf{b}$$

$$x_j \ge 0\ ,\ j = 1, \ldots, n$$

1.3 Linear programming modeling

The first stage in the analysis and solution of a linear programming problem is to formulate the problem by writing a model that represents it. The process of transcribing the verbal description of a problem into a mathematical form that allows the application of linear programming techniques is usually called *modeling*, and it is a particularly difficult aspect. However, it is important, because the solution obtained for the problem will depend on the model that has been formulated. Care must be taken to ensure that the model represents correctly the problem being analyzed. That is why it is worth focusing on the development of the necessary skills to formulate the appropriate models.

In this section, we present a varied collection of problems that are solvable by linear programming techniques. It needs to be reminded that the most important step in formulating a linear model is the proper choice of decision variables. If the decision variables have been properly chosen, the objective function and constraints should follow without much difficulty. When problems arise in determining the objective function and constraints is usually due to an incorrect choice of decision variables.

Example 1. A transportation problem.

A company produces bicycles at three plants in cities C_1, C_2 and C_3. Their production capacity is 1000, 2100 and 1500 bicycles per month, respectively. Four customers, A, B, C and D, from four different locations are demanding 800, 1100, 900 and 1300 bicycles, respectively, every month.

The following table shows unit costs of transporting a bicycle from a given city to a given customer, which may depend on the distance between them.

	Customers			
Cities	A	B	C	D
C_1	10	8	10	13
C_2	19	6	15	16
C_3	14	8	9	6

Formulate a model to find the minimum-cost shipping for the transportation costs given in the table.

- **Decision variables**.

  x_{ij}: number of bicycles transported monthly from city C_i to customer j, $i = 1, 2, 3$, $j = A, B, C, D$.

- **Objective function**: To minimize the transportation costs.

$$\min\ z = 10x_{1A} + 8x_{1B} + 10x_{1C} + 13x_{1D} + 19x_{2A} + 6x_{2B} + 15x_{2C} + \\ +16x_{2D} + 14x_{3A} + 8x_{3B} + 9x_{3C} + 6x_{3D}.$$

- **Constraints**: Supply and demand constraints must be satisfied.

* Supply of production plants: the production capacity.

$$x_{1A} + x_{1B} + x_{1C} + x_{1D} \leq 1000$$
$$x_{2A} + x_{2B} + x_{2C} + x_{2D} \leq 2100$$
$$x_{3A} + x_{3B} + x_{3C} + x_{3D} \leq 1500$$

* It is necessary to satisfy customer's demand.

$$x_{1A} + x_{2A} + x_{3A} \geq 800$$
$$x_{1B} + x_{2B} + x_{3B} \geq 1100$$
$$x_{1C} + x_{2C} + x_{3C} \geq 900$$
$$x_{1D} + x_{2D} + x_{3D} \geq 1300$$

- **The nonnegativity constraints**.

$$x_{ij} \geq 0, \quad i = 1, 2, 3, \quad j = A, B, C, D.$$

Example 2. A production problem.

A firm manufactures three types of pieces, P_1, P_2 and P_3. Three different kind of machines are used in the manufacturing process, A, B and C. The following table shows the number of hours each machine is available for manufacturing and the production cost.

Machine	Availability (hours/week)	Production cost (euro/hour)
A	1000	6
B	1000	4
C	1000	5

Each type of piece needs a different amount of processing time in each of the machines, as can be seen in the following table:

Machine	P_1	P_2	P_3
A	1	2	3
B	2	3	1
C	1	1	1

Two materials are used in the production process, M_1 and M_2. The availability of them is of 1000 kg and 1200 kg, respectively. The following table shows the amount of material needed in the production of one piece of each type:

Piece	M_1(kg/piece)	M_2(kg/piece)
P_1	1	2
P_2	1	3
P_3	3	1

1 kg of material M_1 costs 1.5 euros and 1 kg of material M_2 3 euros. On the other hand, each piece is sold at the price of 50, 56 and 70 euros, respectively. The firm aims to organize the production in order to obtain the maximum benefit from it.

- **Decision variables**.

  x_j: number of pieces P_j that the firm will produce weekly, $j = 1, 2, 3$.

- **Objective function**: To maximize the benefit.

  * Selling price: $= 50x_1 + 56x_2 + 70x_3$.
  * Materials cost: $= (1 \times 1.5 + 2 \times 3)x_1 + (1 \times 1.5 + 3 \times 3)x_2 + (3 \times 1.5 + 1 \times 3)x_3$.
  * Production cost: $= (1 \times 6 + 2 \times 4 + 1 \times 5)x_1 + (2 \times 6 + 3 \times 4 + 1 \times 5)x_2 + (3 \times 6 + 1 \times 4 + 1 \times 5)x_3$.

  The benefit is calculated as follows:

  Benefit = Selling price $-$ Materials cost $-$ Production cost.

  This gives us the following objective function:

$$\max\ z = 23.5x_1 + 16.5x_2 + 35.5x_3.$$

- **Constraints**: The availability of machines and material is constrained.

$$\begin{array}{rrrll} x_1 & +2x_2 & +3x_3 & \leq 1000 & \text{(Machine A)} \\ 2x_1 & +3x_2 & +x_3 & \leq 1000 & \text{(Machine B)} \\ x_1 & +x_2 & +x_3 & \leq 1000 & \text{(Machine C)} \end{array}$$

$$\begin{array}{lllll} x_1 & +x_2 & +3x_3 & \leq 1000 & \text{(Material } M_1) \\ 2x_1 & +3x_2 & +x_3 & \leq 1200 & \text{(Material } M_2) \end{array}$$

- **The nonnegativity constraints**: $x_1, x_2, x_3 \geq 0$.

Example 3. A product-mix problem.

A fuel company produces two types of fuel, A and B, by mixing three types of crude oil. The following table shows the number of crude oil barrels available and the cost of each barrel:

	Availability (barrels)	Cost (units)
Crude oil O_1	2000	10
Crude oil O_2	3000	8
Crude oil O_3	1000	12

The quality of fuels A and B is considered to be acceptable if the crude oil mixture satisfies the following requirements:

- At least 30% of fuel A must be crude oil O_1, at least 20% crude oil O_2 and no more than 30% crude oil O_3.
- At least 25% of the composition of fuel B must be crude oil O_1, at least 25% crude oil O_2, and at least 25% crude oil O_3.

The selling prices of a barrel of fuel A and fuel B are 40 and 35 units, respectively.

The aim is to organize the fuel production in order to obtain the maximum benefit.

- **Decision variables**:

  x_{ij} : The amount of barrels of crude oil O_i in the composition of fuel j, $i = 1, 2, 3$, $j = A, B$.

- **Objective function**: To maximize benefit.

- Selling price: $= 40(x_{1A} + x_{2A} + x_{3A}) + 35(x_{1B} + x_{2B} + x_{3B})$.
- Production cost: $= 10(x_{1A} + x_{1B}) + 8(x_{2A} + x_{2B}) + 12(x_{3A} + x_{3B})$.

The benefit is calculated by operating like that: Selling price $-$ Production cost. We obtain the following objective function:

$$\max z = 30x_{1A} + 32x_{2A} + 28x_{3A} + 25x_{1B} + 27x_{2B} + 23x_{3B}.$$

- **Constraints**: They deal with the amount of crude oil barrels available and the requirements that the mixture must satisfy.

  * Availability of crude oil.

$$\begin{aligned} x_{1A} + x_{1B} &\leq 2000 \quad \text{(Crude oil } O_1) \\ x_{2A} + x_{2B} &\leq 3000 \quad \text{(Crude oil } O_2) \\ x_{3A} + x_{3B} &\leq 1000 \quad \text{(Crude oil } O_3) \end{aligned}$$

  * Requirements of fuels.

$$\begin{aligned} x_{1A} &\geq \frac{30}{100}(x_{1A} + x_{2A} + x_{3A}) \\ x_{2A} &\geq \frac{20}{100}(x_{1A} + x_{2A} + x_{3A}) \\ x_{3A} &\leq \frac{30}{100}(x_{1A} + x_{2A} + x_{3A}) \\ x_{1B} &\geq \frac{25}{100}(x_{1B} + x_{2B} + x_{3B}) \\ x_{2B} &\geq \frac{25}{100}(x_{1B} + x_{2B} + x_{3B}) \\ x_{3B} &\geq \frac{25}{100}(x_{1B} + x_{2B} + x_{3B}) \end{aligned}$$

- **Nonnegativity constraints**:

$$x_{ij} \geq 0, \quad i = 1, 2, 3, \quad j = A, B.$$

Example 4. A diet problem.

A nutrition center wants to prepare a diet that will satisfy the following A, B, C and D vitamin requirements: at least 25 milligrams of vitamin A, between 25 and 30 milligrams of vitamin B, at least 22 milligrams of vitamin C and no more than 17 milligrams of vitamin D.

The following four foods are available for consumption: F_1, F_2, F_3 and F_4. The vitamin content per unit of each food (milligrams per gram) and the cost of a gram of each type of food are summarized in the table:

	Vitamins (mg/g)				Cost
Food	A	B	C	D	(euro/g)
F_1	2	1	0	1	0.014
F_2	1	2	1	2	0.009
F_3	1	0	2	0	0.013
F_4	1	2	1	1	0.016

A linear programming model that can be used to satisfy the nutritional requirements at minimum cost can be formulated in the following way:

- **Decision variables**.

  x_j: grams of each type of food F_j included in the diet, $j = 1, 2, 3, 4$.

- **Objective function**: To minimize the cost of the diet.

$$\min\ z = 0.014x_1 + 0.009x_2 + 0.013x_3 + 0.016x_4.$$

- **Constraints**: To guarantee that the necessary amount of vitamins will be included in the diet.

$$\begin{aligned} 2x_1 + x_2 + x_3 + x_4 &\geq 25 \quad \text{(Vitamin } A\text{)} \\ x_1 + 2x_2 + 2x_4 &\geq 25 \quad \text{(Vitamin } B\text{)} \\ x_1 + 2x_2 + 2x_4 &\leq 30 \quad \text{(Vitamin } B\text{)} \\ x_2 + 2x_3 + x_4 &\geq 22 \quad \text{(Vitamin } C\text{)} \\ x_1 + 2x_2 + x_4 &\leq 17 \quad \text{(Vitamin } D\text{)} \end{aligned}$$

- **Nonnegativity constraints**: $x_1, x_2, x_3, x_4 \geq 0$.

Example 5. A cutting problem.

An enterprise produces 5m long wooden sticks. However, customers demand shorter wooden sticks. In fact, they demand 100 3m long sticks, 100 2m long sticks, 300 1.5m long ones and 150 1m long sticks. Therefore, the enterprise must meet its demands by cutting up its 5m long sticks.

The enterprise must decide how each 5m long stick should be cut. There are different ways to cut them. The enterprise wants to minimize the waste incurred in meeting the customer demands. The following table shows 7 different ways to cut the sticks.

Cutting	Length			
option	3m	2m	1.5m	1m
1	1	1	0	0
2	1	0	0	2
3	0	2	0	1
4	0	1	2	0
5	0	1	0	3
6	0	0	2	2
7	0	0	0	5

- **Decision variables**.

  x_j: number of 5m long sticks cut according to cutting option $j,\ j = 1, \ldots, 7$.

- **Objective function**: To minimize the total amount of 5m long sticks cut.

$$\min\ z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7.$$

- **Constraints**: To meet the customers demands.

$$\begin{aligned} x_1 + x_2 &\geq 100 \\ x_1 + 2x_3 + x_4 + x_5 &\geq 100 \\ 2x_4 + 2x_6 &\geq 300 \\ 2x_2 + x_3 + 3x_5 + 2x_6 + 5x_7 &\geq 150 \end{aligned}$$

- **Nonnegativity constraints**: $x_1, \ldots, x_7 \geq 0$.

The problem can also be formulated by defining some more possible cutting options if we consider that pieces of stick less than half a meter long are an acceptable waste of stick. In the following table the new acceptable cutting options are shown:

Cutting	Length				Waste
option	3m	2m	1.5m	1m	(m)
1	1	1	0	0	0
2	1	0	1	0	0.5
3	1	0	0	2	0
4	0	2	0	1	0
5	0	1	2	0	0
6	0	1	1	1	0.5
7	0	1	0	3	0
8	0	0	3	0	0.5
9	0	0	2	2	0
10	0	0	1	3	0.5
11	0	0	0	5	0

This time there are 11 cutting options and therefore, 11 decision variables will be defined to decide the number of 5m long sticks that will be cut according to each cutting option. The following linear programming model can be used to determine the optimal way to cut sticks:

$$\min\ z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11}$$

subject to

$$\begin{aligned}
x_1 + x_2 + x_3 &\geq 100 \\
x_1 + 2x_4 + x_5 + x_6 + x_7 &\geq 100 \\
x_2 + 2x_5 + x_6 + 3x_8 + 2x_9 + x_{10} &\geq 300 \\
2x_3 + x_4 + x_6 + 3x_7 + 2x_9 + 3x_{10} + 5x_{11} &\geq 150 \\
x_1, \ldots, x_{11} &\geq 0
\end{aligned}$$

1.4 Graphical solution

In general, and even though not all linear problems can be solved graphically, they all can be geometrically interpreted. It is worth to study the graphical solution of linear problems, because it enables to observe graphically important concepts in linear programming, such as the improvement of a solution, types of solutions, extreme points, etc.

The set of solutions or feasible region of a linear inequality system can be graphically illustrated by representing the equation associated with each inequality and determining the half-space that contains the points that satisfy the inequality. By the nonnegativity constraints, the points can only fall in the first quadrant. By proceeding this way, we will obtain the polygon of solutions. The objective function is a family of parallel straight lines, one for each value of z. The line representing the objective function is moved in the optimization direction as much as possible, until the optimal point is reached. If there exists a bounded optimal solution to the problem, then the optimal value for the objective function will be found in an extreme point of the polygon.

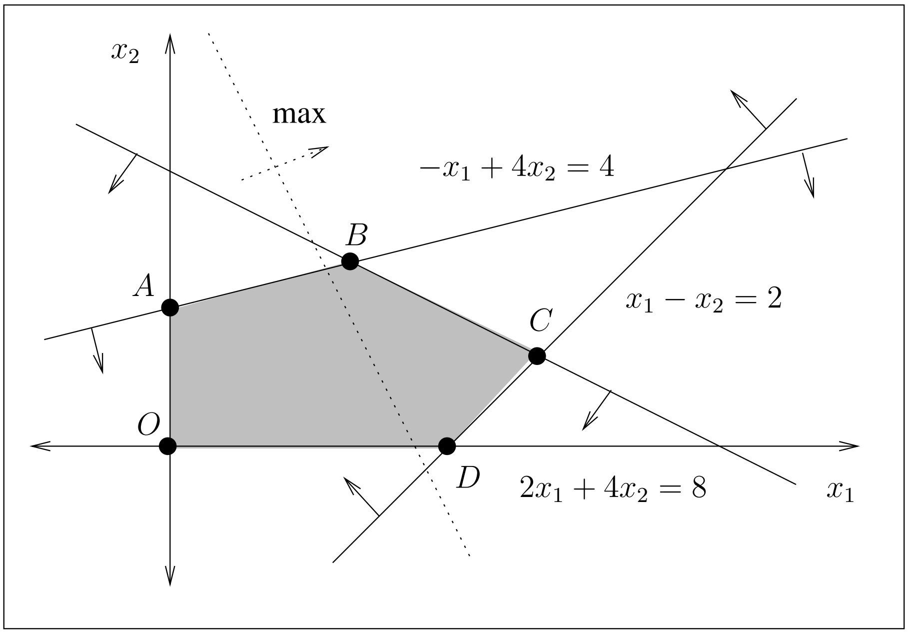
In this section we analyze the graphical solution of some linear models with only two variables.

Example. A problem with a unique optimal solution. Consider the following linear problem:

$$
\begin{aligned}
\max \ z = 6x_1 + 3x_2 \\
\text{subject to} \qquad \\
2x_1 + 4x_2 \leq 8 \\
-x_1 + 4x_2 \leq 4 \\
x_1 - x_2 \leq 2 \\
x_1, x_2 \geq 0
\end{aligned}
$$

The objective is to choose x_1 and x_2 such that they verify the constraints and maximize the objective value $z = 6x_1 + 3x_2$.

We can represent graphically the set of points that satisfy the linear inequalities. Each constraint in the model is a half-space in the plane. For example, in order to represent the set of points satisfying $2x_1 + 4x_2 \leq 8$, we draw the straight line $2x_1 + 4x_2 = 8$. This straight line divides the plane in two half-spaces. The points satisfying the constraint are contained in one of the two half-spaces. We can test whether one point, the origin for instance, satisfies the constraint to decide which one of the two half-spaces contains all the points satisfying the constraint. In the graphical representation, we illustrate it by using small arrows. After representing graphically all the constraints of the problem, including the nonnegativity constraints, we obtain the set of solutions of the problem, which is shown by the shaded region in the following graphical representation:



The polygon $OABCD$ is a convex set. The extreme-points in the convex set can be determined by solving linear equations systems.

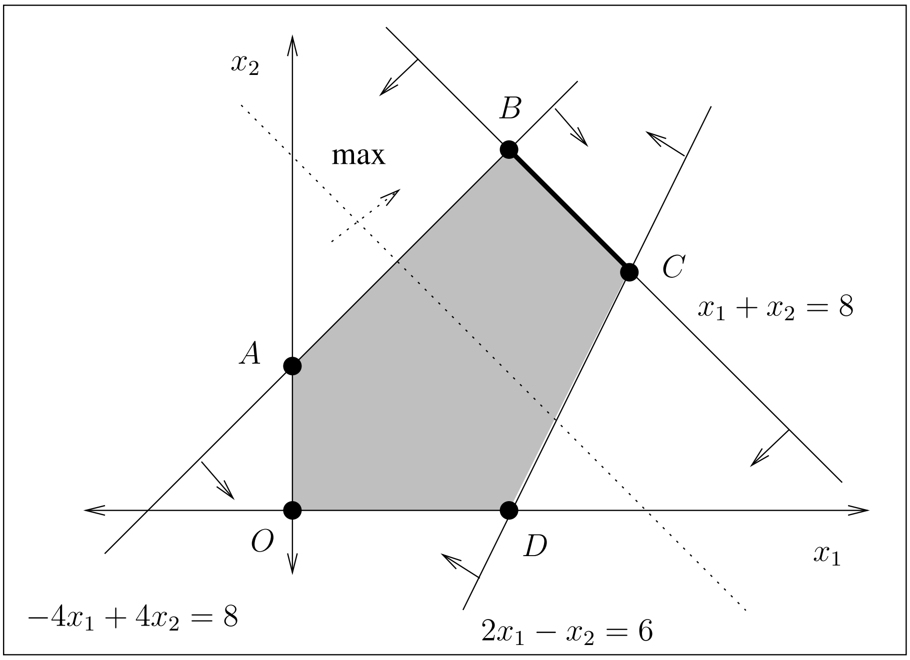
The point O is the origin of the coordinate system. The point $A = (0, 1)$ is the intersection between the straight line $-x_1 + 4x_2 = 4$ and the x_2 axis. The point $D = (2, 0)$ is the intersection between the straight line $x_1 - x_2 = 2$ and the x_1 axis. The point $B = (\frac{4}{3}, \frac{4}{3})$ is the intersection between the straight lines $-x_1 + 4x_2 = 4$ and $2x_1 + 4x_2 = 8$. The point $C = (\frac{8}{3}, \frac{2}{3})$ is the intersection between the straight lines $x_1 - x_2 = 2$ and $2x_1 + 4x_2 = 8$.

We now search for the optimal solution, which will be the point in the feasible region with the largest value of z. To draw the objective function for a particular value of z, we choose any point in the feasible region and compute its z value. We can find all other objective function lines by moving parallel to the line we have drawn. Thus, we move the line that represents the objective function in the direction that increases z. Note that we move the line as long as it intersects with the feasible region. Once the border of the feasible region is reached, the optimal solution is found. The graphical representation shows that the optimal solution is the point C, and that the optimal objective value is $z^* = 18$.

Example. A problem with multiple optimal solutions. Let us consider the following linear model:

$$\begin{aligned} \max\ z &= x_1 + x_2 \\ \text{subject to} & \\ x_1 + x_2 &\leq 8 \\ -4x_1 + 4x_2 &\leq 8 \\ 2x_1 - x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

We proceed as in the previous example to calculate the feasible region. In this case, the set of solutions is the polygon $OABCD$ that appears shaded in the graphical solution. Point B is the intersection between lines $x_1 + x_2 = 8$ and $-4x_1 + 4x_2 = 8$, thus, $B = (3,\ 5)$. Point C is the intersection between lines $x_1 + x_2 = 8$ and $2x_1 - x_2 = 6$, thus, $C = (\frac{14}{3},\ \frac{10}{3})$.

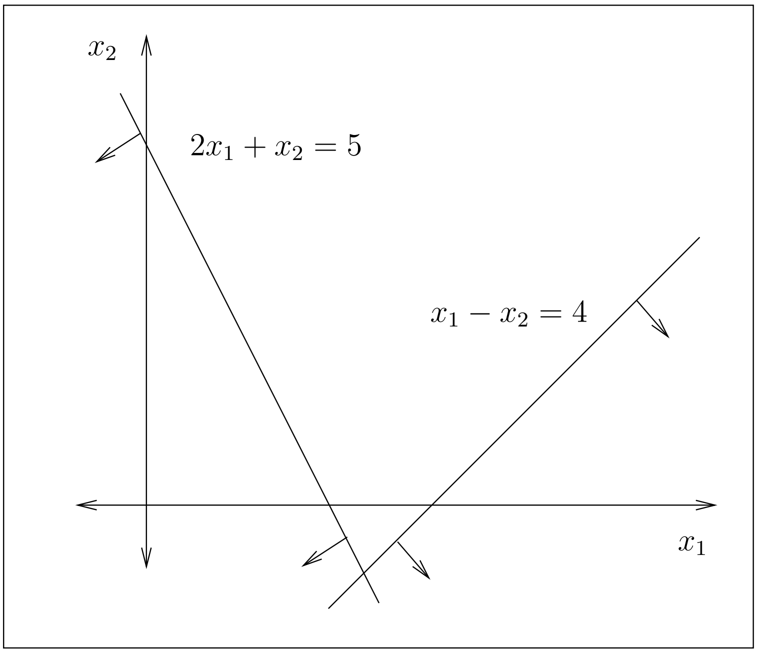


z increases as we move the objective function line in a northeast direction, so the largest value of z in the feasible region occurs at some points on the boundary of the region. Extreme-points B and C, together with all points on the line segment BC are optimal. The optimal objective function value is $z^* = 8$.

Example. An infeasible problem. Consider the following linear model:

$$
\begin{aligned}
\max\ z = x_1 + x_2 \\
\text{subject to} \\
2x_1 + x_2 \leq 5 \\
x_1 - x_2 \geq 4 \\
x_1, x_2 \geq 0
\end{aligned}
$$

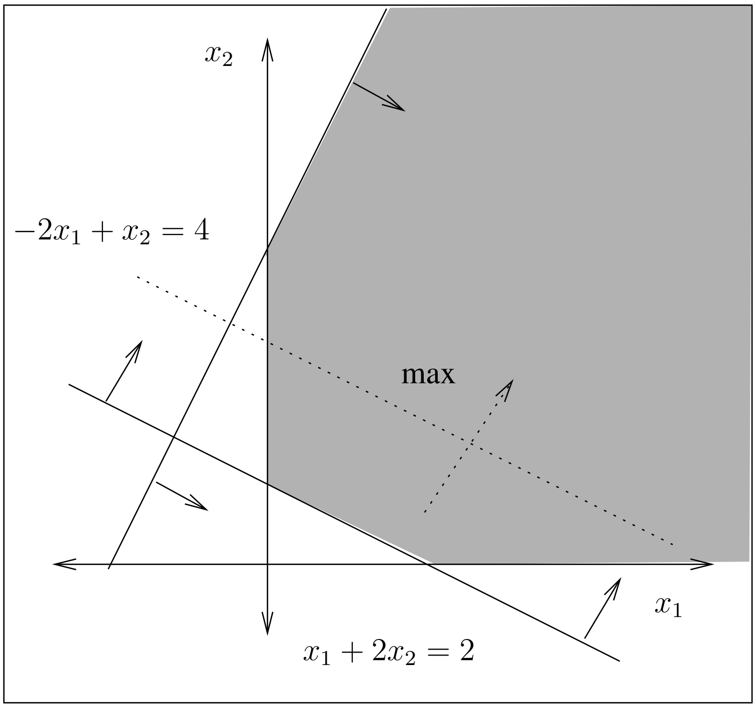
If we represent graphically all the constraints, we can see that the feasible region is empty, which means that no point satisfies all the constraints simultaneously. Therefore, the linear model is said to be infeasible.



Example. An unbounded feasible region. Unbounded solution. Consider the following linear model:

$$\begin{aligned} \max\ z = x_1 + 2x_2 \\ \text{subject to} \\ x_1 + 2x_2 \geq 2 \\ -2x_1 + x_2 \leq 4 \\ x_1, x_2 \geq 0 \end{aligned}$$

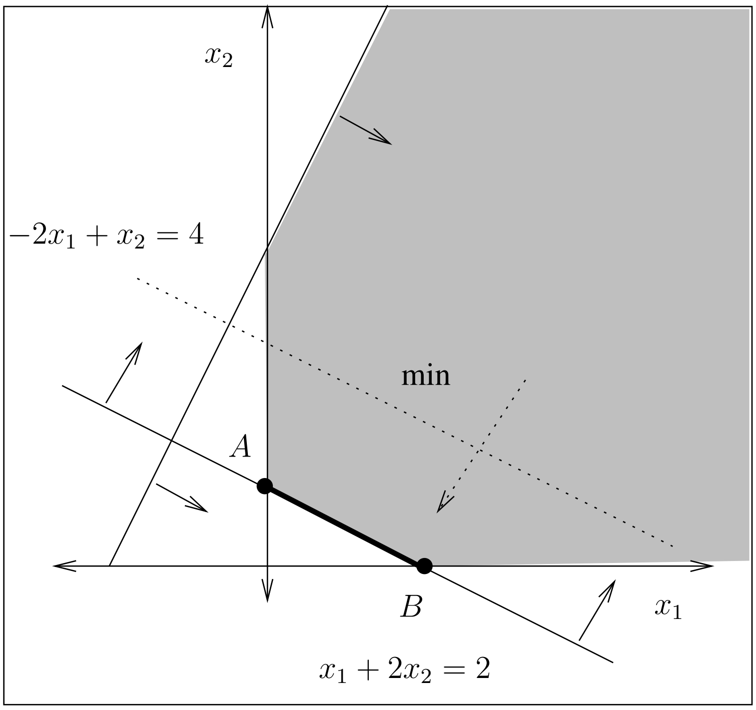
The feasible region for the problem is the shaded unbounded region in the figure. Moving the objective function line in the optimization direction (northeast direction, which makes x_1 and x_2 larger), we see that it will always intersect the feasible region. Therefore, z has an arbitrarily large value. The optimal solution is said to be unbounded.



Example. An unbounded region. A bounded solution. Let us consider this linear model:

$$\begin{aligned} \min\ z = x_1 + 2x_2 \\ \text{subject to} \\ x_1 + 2x_2 \geq 2 \\ -2x_1 + x_2 \leq 4 \\ x_1, x_2 \geq 0 \end{aligned}$$

The feasible region in this example is unbounded, but a bounded solution can be found by moving the objective function line in the optimization direction (southwest direction, which makes x_1 and x_2 smaller). In this case, there are multiple optimal solutions as points $A = (0,\ 1)$, $B = (2,\ 0)$ and the infinite points lying on the segment line AB are optimal, $z^* = 2$.



We have illustrated the different types of solutions that can be found while solving linear problems. We now need to determine the conditions that must hold in order to identify each of the different kinds of solutions. This will be done in Chapter 2, and the simplex algorithm will be introduced to solve linear problems.

In Appendix A, a linear algebra review can be found and definitions and properties about half-spaces, convex sets, etc. are given in more detail. In Chapter 2, we will prove that the optimal solution of a linear model lies on an extreme point of the convex feasible region.